PAPER 4

FUNDAMENTALS OF BUSINESS MATHEMATICS AND STATISTICS (FMS)

FOUNDATION

The Institute of Cost Accountants of India
(Statutory body under an Act of Parliament)
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FOUNDATION : PAPER - 4

FUNDAMENTALS OF BUSINESS MATHEMATICS AND STATISTICS

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STUDY NOTES

The Institute of Cost Accountants of India
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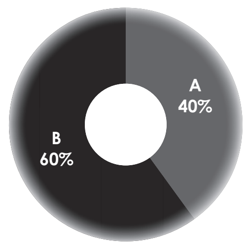
Syllabus

PAPER 4: FUNDAMENTALS OF BUSINESS MATHEMATICS AND STATISTICS (FBMS)

Syllabus Structure

A	Fundamentals of Business Mathematics	40%
B	Fundamentals of Business Statistics	60%



ASSESSMENT STRATEGY

There will be written examination paper of three hours.

OBJECTIVES

To gain understanding on the fundamental concepts of mathematics and statistics and its application in business decision-making

Learning Aims

The syllabus aims to test the student's ability to:

- Understand the basic concepts of basic mathematics and statistics
- Identify reasonableness in the calculation
- Apply the basic concepts as an effective quantitative tool
- Explain and apply mathematical techniques
- Demonstrate to explain the relevance and use of statistical tools for analysis and forecasting

Skill sets required

Level A: Requiring the skill levels of knowledge and comprehension

CONTENTS	
Section A: Fundamentals of Business Mathematics	**40%**
1. Arithmetic	
2. Algebra	
3. Calculus	
Section B: Fundamentals of Business Statistics	**60%**
4. Statistical representation of Data	
5. Measures of Central Tendency and Dispersion	
6. Correlation and Regression	
7. Index Numbers	
8. Time Series Analysis- basic applications including Moving Average	
9. Probability	
10. Theoretical Distribution	

SECTION A: FUNDAMENTALS OF BUSINESS MATHEMATICS [40 MARKS]

1. Arithmetic

(a) Ratios and Proportions

(b) Simple and Compound interest including application of Annuity

(c) Bill Discounting and Average Due Date

(d) Mathematical reasoning – basic application

2. Algebra

(a) Set Theory and simple application of Venn Diagram

(b) Variation, Indices, Logarithms

(c) Permutation and Combinations – basic concepts

(d) Linear Simultaneous Equations (3 variables only)
(e) Quadratic Equations
(f) Solution of Linear inequalities (by geometric method only)
(g) Determinants and Matrices

3. **Calculus**
(a) Constant and variables, Functions, Limit & Continuity
(b) Differentiability & Differentiation, Partial Differentiation
(c) Derivatives – First order and Second order Derivatives
(d) Maxima & Minima – without constraints and with constraints using Lagrange transform
(e) Indefinite Integrals: as primitives, integration by substitution, integration by part
(f) Definite Integrals: evaluation of standard integrals, area under curve

SECTION B: FUNDAMENTALS OF BUSINESS STATISTICS [60 MARKS]

4. **Statistical Representation of Data**
(a) Diagrammatic representation of data
(b) Frequency distribution
(c) Graphical representation of Frequency Distribution – Histogram, Frequency Polygon, Ogive, Pie-chart

5. **Measures of Central Tendency and Dispersion**
(a) Mean, Median, Mode, Mean Deviation
(b) Quartiles and Quartile Deviation
(c) Standard Deviation
(d) Co-efficient of Variation, Coefficient of Quartile Deviation

6. **Correlation and Regression**
(a) Scatter diagram
(b) Karl Pearson's Coefficient of Correlation
(c) Rank Correlation
(d) Regression lines, Regression equations, Regression coefficients

7. **Index Numbers**
(a) Uses of Index Numbers
(b) Problems involved in construction of Index Numbers
(c) Methods of construction of Index Numbers

8. **Time Series Analysis – basic application including Moving Average**
(a) Moving Average Method
(b) Method of Least Squares

9. **Probability**
(a) Independent and dependent events; Mutually exclusive events
(b) Total and Compound Probability; Baye's theorem; Mathematical Expectation

10. **Theoretical Distribution**
(a) Binomial Distribution, Poisson Distribution – basic application
(b) Normal Distribution – basic application

Content

SECTION - A

BUSINESS MATHEMATICS

Study Note 1 : Arithmetic

Study Note 2 : Algebra

Study Note 3 : Calculus

SECTION - B

STATISTICS

Study Note 4 : Statistical Representation of Data

Study Note 5 : Measures Of Central Tendency and Measures of Dispersion

Study Note 6 : Correlation and Regression

Study Note 7 : Index Numbers

Study Note 8 : Time Series Analysis

Study Note 9 : Probability

Study Note 10 : Theoretical Distribution

Section - A
BUSINESS MATHEMATICS

Study Note - 1

ARITHMETIC

This Study Note includes

1.1 Ratio & Proportion

1.2 Simple & Compound Interest (Including Application of Annuity)

1.3 Discounting of Bills & Average Due Date

1.4 Mathematical Reasoning - Basic Application

1.1 RATIO AND PROPORTION

1.1.1 Ratio

The ratio between quantities a and b of same kind is obtained by dividing a by b and is denoted by a : b.

Inverse Ratio: For the ratio a : b inverse ratio is b : a.

A ratio remains unaltered if its terms are multiplied or divided by the same number.

a : b = am : bm (multiplied by m)

$a : b = \frac{a}{m} : \frac{b}{m}$ (divided by m $\neq$ 0)

Thus $2 : 3 = 2 \times 2 : 3 \times 2 = 4 : 6 = \frac{4}{2} : \frac{6}{2} = 2 : 3$

If a = b, the ratio a : b is known as **ratio of equality.**

If a > b, then ratio a : b is known as ratio of **greater inequality** i.e. 7 : 4 And for a < b, ratio a : b will be the ratio of Lesser inequality i.e. 4 : 7.

Solved Examples:

Example 1 : Reduce the two quantities in same unit.

If a = 2kg., b = 400gm, then a : b = 2000 : 400 = 20 : 4 = 5 : 1 (here kg is changed to gm)

Example 2 : If a quantity increases by a given ratio, multiply the quantity by the greater ratio.

If price of crude oil increased by 4 : 5, which was ₹ 20 per unit of then present price = $20 \times \frac{5}{4}$ = ₹ 25 per unit.

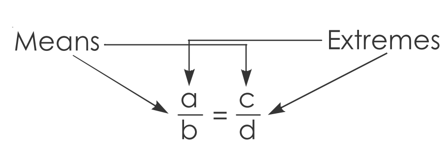
Example 3 : If again a quantity decreases by a given ratio, then multiply the quantity by the lesser ratio.

In the above example of the price of oil is decreased by 4 : 3, the present price = $20 \times \frac{3}{4}$ = ₹ 15 per unit.

If both increase and decrease of a quantity are present is a problem, then multiply the quantity by greater ratio for increase and lesser ratio for decrease, to obtain the final result.

1.1.2 Proportion

An equation that equates two ratios is a proportion. For instance, if the ratio $\frac{a}{b}$ is equal to the ratio $\frac{c}{d}$, then the following proportion can be written:



The numbers a and d are the extremes of the proportion. The numbers b and c are the means of the proportion.

Properties of proportions

1. Cross product property: The product of the extremes equals the product of the means.

   If $\frac{a}{b} = \frac{c}{d}$, then ad = bc.

2. Reciprocal property: If two ratios are equal, then their reciprocals are also equal.

   If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

Few Terms :

1. Continued proportions : The quantities a, b, c, d, e…. are said to be in continued proportion of a : b = b : c = c …. Thus 1, 3, 9, 27, 81, ….. are in continued proportion as 1 : 3 = 3 : 9 = 9 : 27 = 27 : 81 = ….

Say for example : If 2, x and 18 are in continued proportion, find x. Now 2 : x = x : 18 or,

$$\frac{2}{x} = \frac{x}{18} \text{ or}, x^2 = 36 \text{ or}, x = \pm 6$$

Observation: If a, b, c are in continued proportion, the $b^2 = ac, b = \pm\sqrt{ac}$.

2. Compound Proportion : If two or more ratios are multiplied together then they are known as compounded.

Thus $a_1\ a_2\ a_3 : b_1\ b_2\ b_3$ is a compounded ratios of the ratios $a_1 : b_1$; $a_2 : b_2$ and $a_3 : b_3$. This method is also known as **compound rule of three.**

Example 4 : 10 men working 8 hours a day can finish a work in 12 days. In how many days can 12 men working 5 hours a day finish the same work.?

	Men	Hours	day
Arrangement :	10	8	12
	12	5	x

$$x = 12 \times \frac{8}{5} \times \frac{10}{12} = 16 \text{ days}$$

Observation : less working hour means more working days, so multiply by greater ratio $\frac{8}{5}$. Again more men means less number of days, so multiply by lesser ratio $\frac{10}{12}$.

Derived Proportion : Given quantities a, b, c, d are in proportion.

(i) Invertendo : If a : b = c : d then b : a = d : c

(ii) Alternendo : If a : b = c : d, then a : c = b : d

(iii) Componendo and Dividendo

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Proof : Let $\frac{a}{b}=\frac{c}{d}=k$, then a = bk, c = dk

L. H. S. = $\frac{bk+b}{bk-b}=\frac{b(k+1)}{b(k-1)}=\frac{k+1}{k-1}$

R. H. S. = $\frac{dk+d}{dk-d}=\frac{d(k+1)}{d(k-1)}=\frac{k+1}{k-1}$. Hence the result, L.H.S. = R.H.S. (Proved)

Note. 1. $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ is sometimes written as x : y : z = a : b : c.

2. If x : y = a : b, it does not mean x = a, y = b. It is however to take x = ka, y = kb.

Solved Examples :

Example 5 : $\frac{4x-3z}{4c}=\frac{4z-3y}{3b}=\frac{4y-3x}{2a,}$ If , show that each ratio is equal to $\frac{x+y+z}{2a+3b+4c}$.

Each of the given ratio $=\frac{4x-3z+4z-3y+4y-3x}{4c+3b+2a}=\frac{x+y+z}{2a+3b+4c}$

Example 6 : The marks obtained by four examinees are as follows :

A : B = 2 : 3, B : C = 4 : 5, C : D = 7 : 9, find the continued ratio.

A : B = 2 : 3

B : C = 4 : 5 = $4\times\frac{3}{4}:5\times\frac{3}{4}=3:\frac{15}{4}$ [for getting same number in B, we are to multiply by $\frac{3}{4}$]

C : D = 7 : 9 = $7\times\frac{15}{28}:\frac{1}{9}\times\frac{15}{28}=\frac{15}{4}:\frac{135}{28}$ [to same term of C, multiply by $\frac{15}{28}$]

A : B : C : D = $2:3:\frac{15}{4}:\frac{135}{28}=56:84:105:135.$

Example 7 : Two numbers are in the ratio of 3 : 5 and if 10 be subtracted from each of them, the remainders are in the ratio of 1 : 5, find the numbers.

Let the numbers be x and y, so that $\frac{x}{y}=\frac{3}{5}$ or, 5x = 3y...(I)

Again $\frac{x-10}{y-10}=\frac{1}{5}$

or, 5x–y = 40 ….(ii) , Solving (I) & (ii), x = 12, y =20

∴ Required Numbers are 12 and 20.

Example 8 : The ratio of annual incomes of A and B is 4 : 3 and their annual expenditure is 3 : 2 . If each of them saves ₹ 1000 a year, find their annual income.

Let the incomes be 4x and 3x (in ₹)

Now $\frac{4x-1000}{3x-1000}=\frac{3}{2}$ or, x = 1000 (on reduction)

∴ Income of A = ₹ 4000, that of B = ₹ 3000.

Example 9 : The prime cost of an article was three times the value of material used. The cost of raw materials was increased in the ratio 3 : 4 and the productive wage was increased in the ratio 4 : 5. Find the present prime cost of an article, which could formerly be made for ₹ 180.

Prime cost = x + y, where x = productive wage, y = material used.

Now prime cost = 180 =3y or, y = 60, again x + y = 180, x = 180–y = 180–60 = 120

Present material cost = $\frac{4y}{3}$, present wage = $\frac{5x}{4}$,

∴ Present prime cost = $\frac{4\times60}{3}+\frac{5\times120}{4}=80+150=$ ₹ 230.

SELF EXAMINATION QUESTIONS :

1. The ratio of the present age of a father to that of his son is 5 : 3. Ten years hence the ratio would be 3 : 2. Find their present ages. [Ans. 50,30]

2. The monthly salaries of two persons are in the ratio of 3 : 5. If each receives an increase of ₹ 20 in salary, the ratio is altered to 13 : 21. Find the respective salaries. [Ans. ₹ 240, ₹ 400]

3. What must be subtracted from each of the numbers 17, 25, 31, 47 so that the remainders may be in proportion. [Ans. 3]

4. In a certain test, the number of successful candidates was three times than that of unsuccessful candidates. If there had been 16 fewer candidates and if 6 more would have been unsuccessful, the numbers would have been as 2 to 1. Find the number of candidates. [Ans. 136]

5. (i) Monthly incomes to two persons are in ratio of 4 : 5 and their monthly expenditures are in the ratio of 7 : 9. If each saves ₹ 50 a month, find their monthly incomes. [Ans. ₹ 400, ₹500]

   (ii) Monthly incomes of Ram and Rahim are in the ratio 5 : 7 and their monthly expenditures are in the ratio 7 : 11. If each of them saves ₹ 60 per month. Find their monthly income.

   [Ans. 200, ₹ 280]

   hints $\frac{5x-60}{7x-60}=\frac{7}{11}$ & etc.

6. A certain product C is made or two ingredients A and B in the proportion of 2 : 5. The price of A is three times that of B.

   The overall cost of C is ₹ 5.20 per tonne including labour charges of 80 paise per tonne. Find the cost A and B per tonne.

   [Ans. ₹ 8.40, ₹ 2.80]

7. The prime cost of an article was three times than the value of materials used. The cost of raw materials increases in the ratio of 3 : 7 and productive wages as 4 : 9. Find the present prime cost of an article which could formerly be made for ₹ 18. [Ans. ₹ 41]

8. There has been increment in the wages of labourers in a factory in the ratio of 22 : 25, but there has also been a reduction in the number of labourers in the ratio of 15 : 11. Find out in what ratio the total wage bill of the factory would be increased or decreased. [Ans. 6 : 5 decrease]

9. Three spheres of diameters 2, 3 and 4 cms. respectively formed into a single sphere. Find the diameter of the new sphere assuming that the volume of a sphere is proportional to the cube of its diameter. [Ans. $\sqrt[3]{81}$ cm]

OBJECTIVE QUESTIONS :

1. Find the ratio compounded of 3 :7, 21 : 25, 50 : 54 [Ans. 1 : 3]
2. What number is to be added to each term of the ratio 2 : 5 to make to equal 4 :5. [Ans. 10]
3. Find the value of x when x is a mean proportional between : (i) x–2 and x+6
   (ii) 2 and 32 [Ans. (i) 3 (ii) ± 8]
4. If the mean proportional between x and 2 is 4, find x [Ans. 8]
5. If the two numbers 20 and x + 2 are in the ratio of 2 : 3 ; find x [Ans. 28]
6. If $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{1}{2}$ find $\frac{a}{b}$ [Ans.9]
7. If 3, x and 27 are in continued proportion, find x [Ans. ± 9]
8. What number is to be added to each term of the ratio 2 : 5 to make it 3 : 4 ? [Ans. 7]

   Hints : $\frac{2+x}{5+x}=\frac{3}{4}$ & etc.

9. If $\frac{a+b}{a-b}=2$ find the value of $\frac{a^2-ab+b^2}{a^2+ab+b^2}$ [Ans. $\frac{7}{13}$]
10. If $\frac{4x-3y}{4c}=\frac{4z-3y}{3b}=\frac{4y-3x}{2a}$ show that each ratio is equal to $\frac{x+y+z}{2a+3b+4c}$

    Hints : each ratio $=\frac{4x-3z+4z-3y+4y-3x}{4c+3b+2a}$ & etc.

11. The ratio of the present age of mother to her daughter is 5 : 3. Ten years hence the ratio would be 3 : 2. Find their present ages. [Ans. 50; 30 years]
12. If A : B = 2 : 3, B : C = 4 : 5 and A : C [Ans. 8 :15]
13. If x : y = 3 : 2, find the value of (4x–2y) : (x + y) [Ans. 8 : 5]
14. If 15 men working 10 days earn ₹ 500. How much will 12 men earn working 14 days? [Ans. ₹ 560]
15. Fill up the gaps : $\frac{a}{b}=\frac{a^2}{-}=\frac{1}{-}=\frac{-}{1}=\frac{b^2}{-}$ [Ans. ab, b/a, a/b, b^3/a, in order]
16. If x, 12, y and 27 are in continued proportion, find the value of x and y [Ans. 8 ; 18]
17. If $\frac{x}{y}=\frac{3}{4}$, find the value of $\frac{7x-4y}{3x+y}$ [Ans. $\frac{5}{13}$]

18. What number should be subtracted from each of the numbers 17, 25, 31, 47 so that the remainders are in proportion. [Ans. 3]

Hints : $\frac{17-x}{25-x}=\frac{31-x}{47-x}$ or $=3$

19. 10 years before, the ages of father and son was in the ratio 5 : 2; at present their total age is 90 years. Find the present age of the son. [Ans. 30 years]

20. The ratio of work done by (x–1) men in (x+1) days to that of (x+2) men in (x–1) days is 9 : 10 , find the value of x.

Hints: $\frac{(x-1)(x+1)}{(x+2)(x-1)}=\frac{9}{10}$ & etc. [Ans. 8]

1.2 SIMPLE & COMPOUND INTEREST (Including Application of Annuity)

1.2.1 Simple Interest

The price to be paid for the use of a certain amount of money (called principal) for a certain period is known as *Interest*. The interest is payable yearly, half-yearly, quarterly or monthly.

The sum of the principal and interest due at any time, is called the *Amount* at that time.

The rate of interest is the interest charged on one unit of principal for one year and is denoted by **i**. If the principal is ₹ 100 then the interest charged for one year is usually called the amount of interest per annum, and is denoted by r (= Pi).

e.g. if the principal is ₹ 100 and the interest ₹ 3, then we say usually that the rate of interest is 3 percent per annum (or r = 3 %)

Here $i=\frac{3}{100}=0.03$ (i.e. interest for 1 rupee for one year).

Simple interest is calculated always on the original principal for the total period for which the sum (principal) is used.

Let P be the principal (original)

n be the number of years for which the principal is used

r be the rate of interest p.a.

I be the amount of interest

i be the rate of interest per unit (i.e. interest on Re. 1 for one year)]

Now I = P.i.n, where $i=\frac{r}{100}$

Amount A = P + I = P + P. i. n = P (1+ i.n) i.e. **A = P (1 + n .i)**

Observation. So here we find four unknown A, P, i., n, out of which if any three are known, the fourth one can be calculated.

SOLVED EXAMPLES :

Example 10 : Amit deposited ₹ 1200 to a bank at 9% interest p.a. find the total interest that he will get at the end of 3 years.

Here P = 1200, $i = \frac{9}{100} = 0.09$, n = 3, I = ?

I = P. i.n = 1200 x 0.09 x 3 = 324.

Amit will get ₹ 324 as interest.

Example 11 : Sumit borrowed ₹ 7500 at 14.5% p.a. for $2\frac{1}{2}$ years. Find the amount he had to pay after that period.

$P = 7500,\ \ i = \frac{14.5}{100} = 0.145,\ \ n = 2\frac{1}{2} = 2.5, A = ?$

A = P (1+ in) = 7500 (1+ 0.145 x 2.5) = 7500 (1+0.3625)

= 7500 × 1.3625 = 10218.75

Reqd. amount = ₹ 10218.75.

Example 12 : Find the simple interest on ₹ 5600 at 12% p.a. from July 15 to September 26, 2013.

Time = number of days from July 15 to Sept. 26

= 16 (July) + 31 (Aug.) + 26 (Sept.)= 73 days.

$P = 5600,\ i = \frac{12}{100} = 0.12,\ n = \frac{73}{365}\text{yr.} = \frac{1}{5}\text{yr.}$

S.I. = P. i.n. = 5600 x 0.12 x $\frac{1}{5}$ =134.4

∴ Reqd. S.I. = ₹ 134.40.

(In counting days one of two extreme days is to be excluded, Usually the first day is excluded).

To find Principle :

Example 13 : What sum of money will amount to ₹ 1380 in 3 years at 5% p.a. simple interest?

Here A = 1380, n = 3, $i = \frac{5}{100} = 0.05, P = ?$

From A = P (1 + 0.05x3) or, 1380 = P (1+0.15)

Or, 1380 = P (1.15) or, $P = \frac{1380}{1.15} = 1200$

∴ Reqd. sum = ₹ 1200

Example 14 : What sum of money will yield ₹ 1407 as interest in $1\frac{1}{2}$ year at 14% p.a. simple interest.

Here S.I. = 1407, n = 1.5, I = 0.14, P = ?

S.I. = P. i.n. or, 1407 = P x 0.14 x 1.5

Or, $P = \frac{1407}{0.14 \times 1.5} = \frac{1407}{0.21} = 6700$

∴ Reqd. amount = ₹ 6700.

Example 15 : What principal will produce ₹ 50.50 interest in 2 years at 5% p.a. simple interest.

S.I. = 50.50, n = 2, i = 0.05, P = ?

S.I. = P. i.n. or, 50.50 = P x 0.05 x 2 = P x 0.10

or, $P = \frac{50.50}{0.10} = 505$

∴ Reqd. principal = ₹ 505.

Problems to find rate % :

Example 16 : A sum of ₹ 1200 was lent out for 2 years at S. I. The lender got ₹ 1536 in all. Find the rate of interest p.a.

P = 1200, A = 1536, n = 2, i = ?

A = P (1 + ni) or 1536 = 1200 (1 +2i) = 1200 + 2400 i

or, 2400i = 1536 – 1200 = 336 or, $i = \frac{336}{2400} = 0.14$

∴ Reqd. rate = 0.14 x 100 = 14%.

Example 17 : At what rate percent will a sum, become double of itself in $5\frac{1}{2}$ years at simple interest?

A = 2P, P = Principal, $n = 5\frac{1}{2}$, i = ?

A = P (1 + ni) or, 2P = P $(1+\frac{11}{2}\ i)$

or, $2 = 1+\frac{11}{2}\ i$ or, $i = \frac{2}{11}$

or, $n = \frac{2}{11} \times 100 = 18.18$ (approx); Reqd. rate = 18.18%.

Problems to find time :

Example 18 : In how many years will a sum be double of itself at 10% p.a. simple interest.

A = 2P, P = Principal, $i = \frac{10}{100}$ = 0.10, n= ?

A = P (1 + ni) or, 2P = P [1 + n(.10)] or, 2 = 1 + n (.10)

or, n (.10) = 1 or, $n = \frac{1}{.10} = 10$

∴ Reqd. time = 10 years.

Example 19 : In what time ₹ 5000 will yield ₹ 1100 @ $5\frac{1}{2}$%?

P = 5000, S.I. = 1100, $i = \frac{5\frac{1}{2}}{100} = 0.055$, n = ?

S.I.= P. ni or, 1100 = 5000 × n × (0.055) = 275 n

or, $n = \frac{1100}{275} = 4.$ ∴ Reqd. time = 4 years.

Example 20 : In a certain time ₹ 1200 becomes ₹ 1560 at 10% p.a. simple interest. Find the principal that will become ₹ 2232 at 8% p.a. in the same time.

In 1st case : P = 1200, A = 1560, i = 0.10, n = ?

1560 = 1200 [1+n (.10)] = 1200 + 120 n

or, 120 n = 360 or, n = 3

In 2nd case : A = 2232, n = 3, i = 0.08, P = ?

2232 = P (1+3×0.08) = P (1 + 0.24) = 1.24 P

or, $P = \frac{2232}{1.24} = 1800.$

Example 21 : A person borrowed ₹ 8,000 at a certain rate of interest for 2 years and then ₹ 10,000 at 1% lower than the first. In all he paid ₹ 2500 as interest in 3 years. Find the two rates at which he borrowed the amount.

Let the rate of interest = r, so that in the 2nd case, rate of interest will be (r–1). Now

$$8000 \times \frac{r}{100} \times 2 + 10,000 \times \frac{(r-1)}{100} \times 1 = 2500$$

or, 160r + 100 r – 100 = 2500 or, r = 10

In 1^{st} case rate of interest = 10% and in 2^{nd} case rate of interest = (10 – 1) = 9%

Calculations of interest on deposits in a bank : Banks allow interest at a fixed rate on deposits from a fixed day of each month up to last day of the month. Again interest may also be calculated by days.

SELF EXAMINATION QUESTIONS

1. What sum will amount to ₹ 5,200 in 6 years at the same rate of simple interest at which ₹ 1,706 amount to ₹ 3,412 in 20 years? [Ans. ₹ 4000]
2. The simple interest on a sum of money at the end of 8 years is $\frac{2}{5}$th of the sum itself. Find the rate percent p.a. [Ans. 5%]
3. A sum of money becomes double in 20 years at simple interest. In how many years will it be triple? [Ans. 40 yrs.]
4. At what simple interest rate percent per annum a sum of money will be double of it self in 25 years? [Ans. 4%]
5. A certain sum of money at simple interest amounts to ₹ 560 in 3 years and to ₹ 600 in 5 years. Find the principal and the rate of interest. [Ans. ₹ 500; 4%]
6. A tradesman marks his goods with two prices, one for ready money and the other for 6 month's credit. What ratio should two prices bear to each other, allowing 5% simple interest. [Ans. 40 : 41]
7. A man lends ₹ 1800 to two persons at the rate of 4% and $4\frac{1}{2}$% simple interest p.a. respectively. At the end of 6 years, he receives ₹ 462 from them. How much did he lend to each other? [Ans. ₹ 800; ₹ 1000]

8. A man takes a loan of ₹ 10,000 at the rate of 6% S.I. with the understanding that it will be repaid with interest in 20 equal annual instalments, at the end of every year. How much he is to pay in each instalment? [Ans. 700.64]

   [Hints. 10,000 (1+20 ×.06) = P (1+19×.06) + P(1+18×.06) + …P]

9. The price of a watch is ₹ 500 cash or it may be paid for by 5 equal monthly instalments of ₹ 110 each, the first instalment to be paid one month after purchase. Find the rate of interest charged.

   [Ans. $42\frac{6}{7}$%]

10. Divide ₹ 12,000 in two parts so that the interest on one part at 12.5% for 4 years is equal to the interest on the second part at 10% for 3 years. [Ans. ₹ 4500; ₹ 7500]

11. Alok borrowed ₹ 7500 at a certain rate for 2 years and ₹ 6000 at 1% higher rate than the first for 1 year. For the period he paid ₹ 2580 as interest in all. Find the two of interest. [Ans. 12%; 13%]

12. If the simple interest on ₹ 1800 exceeds the interest on ₹ 1650 in 3 years by ₹ 45, find the rate of interest p.a. [Ans. 10%]

Objective Question

1. At what rate of S.I. will ₹ 1000 amount to ₹ 1200 in 2 years? [Ans. 10%]
2. In what time will ₹ 2000 amount to ₹ 2600 at 5% S.I.? [Ans. 6 yrs.]
3. At what rate per percent will S.I. on ₹ 956 amount to ₹ 119.50 in $2\frac{1}{2}$ years? [Ans. 5%]
4. To repay a sum of money borrowed 5 months earlier a man agreed to pay ₹ 529.75. Find the amount borrowed it the rate of interest charged was $4\frac{1}{2}$% p.a. [Ans. ₹ 520]
5. What sum of money will amount to ₹ 5200 in 6 years at the same rate of interest (simple) at which ₹ 1706 amount to ₹ 3412 in 20 years? [Ans. ₹ 4000]
6. A sum money becomes double in 20 years at S.I., in how money years will it be triple? [Ans. 40 years]
7. A certain sum of money at S.I. amount to ₹ 560 in 3 years and to ₹ 600 in 5 years. Find the principal and the rate of interest. [Ans. ₹ 500; 4%]
8. In what time will be the S.I. on ₹ 900 at 6% be equal to S.I. on ₹ 540 for 8 years at 5%. [Ans. 4 years]
9. Due to fall in rate of interest from 12% to $10\frac{1}{2}$% p.a.; a money lender's yearly income diminishes by ₹ 90. Find the capital. [Ans. ₹ 6000]
10. A sum was put at S.I. at a certain rate for 2 years. Had it been put at 2% higher rate, it would have fetched ₹ 100 more. Find the sum. [Ans. ₹ 10,000]
11. Complete the S. I. on ₹ 5700 for 2 years at 2.5% p.a. [Ans. ₹ 285]
12. What principal will be increased to ₹ 4600 after 3 years at the rate of 5% p.a. simple interest? [Ans. ₹ 4000]
13. At what rate per annum will a sum of money double itself in 10 years with simple interest ? [Ans. 10%]

15. The simple interest on ₹ 300 at the rate of 4% p.a. with that on ₹ 500 at the rate of 3% p.a. both for the same period, is ₹ 162. Find the time period. [Ans. 6 years]

16. Calculate the interest on ₹ 10,000 for 10 years at 10% p.a. [Ans. ₹ 10,000]

1.2.2 COMPOUND INTEREST

Interest as soon as it is due after a certain period, is added to the principal and the interest for the succeeding period is based upon the principal and interest added together. Hence the principal does not remain same, but increases at the end of each interest period.

A year is generally taken as the interest-period, but in most cases it may be half-year or quarter-year.

Note. Compound interest is calculated by deducting the principal from the amount (principal + interest) at the end of the given period.

Simple Interest for 3 years on an amount was ₹ 3000 and compound interest on the same amount at the same rate of interest for 2 years was ₹ 2, 100. Find the principal and the rate in interest.

Let the amount be ₹ x and rate of interest = x i. Now x. i. 3. = 3000 … (i)

Again $x(1 + i)^2 - x = 2100$, from $C.I = P(1 + i)^2 - P$, P = Principal

or, xi (2 + i) = 2100

or, 1000 (2 + i) = 2100, by (i)

or, (2 + i) = 2.1 or, i = 2.1 – 2 = 0.1 = 10%

From x. (0. 1) . 3 = 3000 or, x = 10,000

∴ Required principal is ₹ 10,000 and rate of interest is 10%

Using Logarithm :

In calculating compound interest for a large number of years, arithmetical calculation becomes too big and out of control. Hence by applying logarithm to the formula of compound interest, the solution becomes easy.

Symbols :

Let P be the Principal (original)

A be the amount

i be the Interest on Re. 1 for 1 year

n be the Number of years (interest period).

Formula : $\mathbf{A = P(1 + i)^n}$ …. (i)

Cor.1. In formula (i) since P amount to A in years, P may be said to be present value of the sum A due in n years.

$$P = \frac{A}{(1+i)^n} = A(1+i)^{-n}$$

Cor.2. Formula (1) may be written as follows by using logarithm :

$$\log A = \log P + n \log (1 + i)$$

Note. If any three of the four unknowns A, P, n and i are given, we can find the fourth unknown from the above formula.

FEW FORMULAE :

Compound Interest may be paid half-yearly, quarterly, monthly instead of a year. In these cases difference in formulae are shown below :

(Taken P = principal, A = amount, T = total interest, i = interest on Re. 1 for 1 year, n = number of years.)

	Time	Amount	I = A – P
(i)	Annual	$A= P(1 + I)^n$	$I = P \{(1 + i)^n - 1\}$
(ii)	Half-Yearly	$A = P\left(1+\frac{i}{2}\right)^{2n}$	$I = P\left[\left(1+\frac{i}{2}\right)^{2n} - 1\right]$
(iii)	Quarterly	$A = P\left(1+\frac{i}{4}\right)^{4n}$	$I = P\left[\left(1+\frac{i}{4}\right)^{4n} - 1\right]$

In general if C.I. is paid p times in a year, then $A = P\left(1+\frac{i}{p}\right)^{pn}$.

i.e. : Let P = ₹ 1000, r = 5% i.e., i = 0.05, n = 24 yrs.

If interest is payable yearly the $A = 1000\ (1 + 0.5)^{24}$

If int. is payable half-yearly the A = $1000\left(1+\frac{0.05}{2}\right)^{2\times24}$

If int. is payable quarterly then A = $1000\left(1+\frac{0.05}{4}\right)^{4\times24}$

Note. or r = 100 I = interest per hundred.

If r = 6% then If, however i = 0.02 then, r = 100 × 0.02 = 2%.

SOLVED EXAMPLES. (using log tables)

[To find C.I.]

Example 22 : Find the compound interest on ₹ 1,000 for 4 years at 5% p.a.

Here P = ₹ 1000, n = 4, i = 0.05, A = ?

We have $A = P\ (1 + i)^n$

$A = 1000\ (1+ 0.05)^4$

Or log A = log 1000 + 4log (1 + 0.05) = 3 + 4 log (1. 05) = 3 + 4 (0.0212) = 3 + 0.0848 = 3.0848

A = antilog 3.0848 = 1215

C.I. = ₹ 1215 – ₹ 1000 = ₹ 215

[To find time]

Example 23 : In what time will a sum of money double itself at 5% p.a. C.I.

Here, P = P, A = 2P, i = 0.05, n = ?

$A = P(1+i)^n$ or, $2P = P(1+0.05)^n = P(1.05)^n$

or, $2 = (1.05)^n$ or $\log 2 = n \log 1.05$

$$\therefore n = \frac{\log 2}{\log 1.05} = \frac{0.3010}{0.0212} = 14.2 \text{ years (Approx)}$$

∴ (anti-logarithm table is not required for finding time).

[To find sum]

Example 24 : The difference between simple and compound interest on a sum put out for 5 years at 3% was ₹ 46.80. Find the sum.

Let P = 100, i = .03, n = 5. From $A = P(1+i)^n$,

$A = 100(1+.03)^5 = 100(1.03)^5$

log A = log 100 + 5 log (1.03) = 2 + 5 (.0128) = 2 + .0640 = 2.0640

∴ A = antilog 2.0640 = 115.9 ∴ C.I. = 115.9 – 100 = 15.9

Again S.I. = 3 × 5 = 15. ∴ difference 15.9 – 15 = 0.9

Diff.	Capital
0.9	100
46.80	x

$$x = 100 \times \frac{46.80}{0.9} = 5,200$$

∴ original sum = ₹ 5,200.

[To find present value]

Example 25 : What is the present value of ₹ 1,000 due in 2 years at 5% compound interest, according as the interest is paid (a) yearly, (b) half-yearly ?

(a) Here A = ₹ 1,000, $i = \frac{5}{100} = 0.05$, n = 2, P = ?

$A = P(1+i)^n$ or $1000 = P(1+.05)^2 = P(1.05)^2$

$$\therefore P = \frac{1000}{(1.05)^2} = \frac{1000}{1.1025} = 907.03$$

∴ Present value = ₹ 907.03

(b) Interest per unit per half-year $\frac{1}{2} \times 0.05 = 0.025$

From $A = P\left(1+\frac{i}{2}\right)^{2n}$ we find.

$$1,000 = P\left(1+\frac{0.05}{2}\right)^{2\times 2} = P(1+.025)^4 = P(1.025)^4$$

or, $P = \frac{1000}{(1.025)^4}$

$\therefore$ log P = log 1000 – 4log (1.025) = 3 – 4 (0.0107) = 3 – 0.0428 = 2.9572

$\therefore$ P = antilog 2.9572 = 906.1.

Hence the present amount = ₹ 906.10

[To find rate of interest]

Example 26 : A sum of money invested at C.I. payable yearly amounts to ₹ 10, 816 at the end of the second year and to ₹ 11,248.64 at the end of the third year. Find the rate of interest and the sum.

Here A_1 = 10,816, n = 2, and A_2 = 11,248.64, n = 3

From A = P (1 + i)^n we get,

10,816 = P (1 + i)^2 … (i) and 11,248.64 = P(1 + i)^3 … (ii)

Dividing (ii) by (i) $\frac{11,248.64}{10,816} = \frac{P(1+i)^3}{P(1+i)^2}$ or, $(1+i) = \frac{11,248.64}{10,816}$

or $i = \frac{11,248.64}{10,816} - 1 = \frac{432.64}{10,816} = .04$, r = i × 100 = .04 × 100 = 4 $\therefore$ reqd. rate = 4%

Now from (I) $P = \frac{10,816}{(1+.04)^2} = \frac{10,816}{(1.04)^2}$

Log P = log 10,816 – 2log (1.04) = 4.034 – 2 (0.170) = 4.034 – .0340 = 4.000

$\therefore$ P = antilog 4.000 = 10,000 $\therefore$ required sum = ₹ 10,000.

SELF EXAMINATION QUESTIONS

1. The difference between the simple interest and compound interest on a sum put out for 2 years at 5% was ₹ 6.90. Find the sum. [Ans. ₹ 2,300]

2. Find the CI. on ₹ 6,950 for 3 years if the interest is payable half-yearly, the rate for the first two years being 6% p.a. and for the third year 9% p.a. [Ans. ₹ 1,589]

3. In what time will a sum of money double itself at 5% C.I. payable half-yearly ? [Ans. 14.01 yrs.]

4. What is the rate per cent p.a. if ₹ 600 amount to ₹ 10,000 in 15 years, interest being compounded half-yearly. [Ans. 19.6%]

5. What is the present value of an investment of ₹ 2,000 due in 6 years with 5% interest compounded semi-annually? [Ans. ₹ 1,488]

6. A sum of ₹ 1000 is invested for 5 years at 12% interest per year. What is the simple interest? If the same amount had been invested for the same period at 10% compound interest per year, how much more interest would he get? [Ans. ₹ 162]

7. A certain sum is invested in a firm at 4% C.I. The interest for the second year is ₹ 25. Find interest for the 3 rd year. [Ans. ₹ 26]

8. The interest on a sum of money invested at compound interest is ₹ 66.55 for the second year and ₹ 72 for the fourth year. Find the principal and rate per cent. [Ans. 1,600 ; 4%]

9. Determine the time period during which a sum of ₹ 1,234 amounts to ₹ 5,678 at 8% p.a. compound interest, payable quarterly. (given log 1234 = 3.0913, log 5678 = 3.7542 and log 1.02 = 0.0086]
[Ans. 19.3 yrs. (approx)]

[**hints :** $5678 = 1234\left(1+\frac{0.08}{4}\right)^{4n}$ & etc.]

10. Determine the time period by which a sum of money would be three times of itself at 8% p. a. C.I. (given log 3= 0.4771, $\log_{10} 1.08 = 0.0334$) [Ans. 14.3 yrs. (approx)]

11. The wear and tear of a machine is taken each year to be one-tenth of the value at the beginning of the year for the first ten years and one-fifteenth each year for the next five years. Find its scrap value after 15 years. [Ans. 24.66%]

12. A machine depreciates at the rate of 10% p.a. of its value at the beginning of a year. The machine was purchased for ₹ 44,000 and the scrap value realised when sold was ₹ 25981.56. Find the number of years the machine was used. [Ans. 5 years (approx)]

1.2.3 ANNUITIES

Definition :

An annuity is a fixed sum paid at regular intervals under certain conditions. The interval may be either a year or a half-year or, a quarter year or a month.

Definition : Amount of an annuity :

Amount of an annuity is the total of all the instalments left unpaid together with the compound interest of each payment for the period it remains unpaid.

Formula :

(i) $A = \frac{P}{i}\left\{(1+i)^n - 1\right\}$ Where A = total(s) amount after n years,

i = rate of interest per rupee per annum.

p = yearly annuity

(ii) If an annuity is payable half-yearly and interest is also compounded half-yearly, then amount A is given by

$$A = \frac{2P}{i}\left[\left(1+\frac{i}{2}\right)^{2n} - 1\right]$$

(iii) If an annuity is payable quarterly and interest is also compounded quarterly, then amount A is given by

$$A = \frac{4P}{i}\left[\left(1+\frac{i}{4}\right)^{4n} - 1\right]$$

Present value of an annuity :

Definition : Present value of an annuity is the sum of the present values of all payments (or instalments) made at successive annuity periods.

Formula :

(i) The present value V of an annuity P to continue for n years is given by

$$V = \frac{P}{i}\left\{1-(1+i)^{-n}\right\}$$ Where i = interest per rupee per annum.

(ii) The Present value V of an annuity P payable half-yearly, then

$$V = \frac{2P}{i}\left[1-\left(1+\frac{i}{2}\right)^{-2n}\right]$$

(ii) The Present value V of an annuity P payable quarterly, then

$$V = \frac{4P}{i}\left[1-\left(1+\frac{i}{4}\right)^{-4n}\right]$$

SOLVED EXAMPLES :

Example 27 :

A man decides to deposit ₹ 20,000 at the end of each year in a bank which pays 10% p.a. compound interest. If the instalments are allowed to accumulate, what will be the total accumulation at the end of 9 years?

Solution :

Let ₹ A be the total accumulation at the end of 9 years. Then we have

$$A = \frac{P}{i}\left\{(1+i)^{n}-1\right\}$$

Here P = ₹ 20,000, $i = \frac{10}{100} = 0.1$, n = 9 years.

$$\therefore A = \frac{20{,}000}{0.1}\left\{(1+.1)^{9}-1\right\} = 2{,}00{,}000\left\{(1.1)^{9}-1\right\} = 2{,}00{,}000\ (2.3579-1)$$

= 2,00,000 x 1.3579 = ₹ 2,71,590

∴ The required total accumulation = ₹ 2,71,590.

Example 28 :

A truck is purchased on instalment basis, such that ₹ 10,000 is to be paid on the signing of the contract and five yearly instalments of ₹ 5,000 each payable at the end of 1st, 2nd, 3rd, 4th and 5th years. If interest is charged at 10% per annum what would be the cash down price?

Solution :

Let V be the present value of the annuity of ₹ 5,000 for 5 years at 10% p.a. compound interest, then cash down price of the truck is ₹ (10,000 + V).

Now, $V = \frac{p}{i}\left\{1-(1+i)^{-n}\right\}$. Here, P = 5,000, n = 5, $i = \frac{10}{100} = 0.1$

$$\therefore V = \frac{5{,}000}{0.1}\left\{1-(1.1)^{-5}\right\} = 50{,}000\ \frac{(1.1)^{5}-1}{(1.1)^{5}} = 50{,}000\ \frac{1.61051-1}{1.61051}$$

$$= 50{,}000 \times \frac{0.61051}{1.61051} = 50{,}000 \times 0.379079$$

Hence the required cash down price of the truck = ₹ (18,953.95 + 10,000)

= ₹ 28,953.95

Example 29 :

The accumulation in a Providend Fund are invested at the end of every year to year 11% p.a. A person contributed 15% of his salary to which his employer adds 10% every month. Find how much the accumulations will amount to at the end of 30 years for every 100 rupees of his monthly salary.

Solution :

Let the monthly salary of the person be ₹ Q, then the total monthly contribution to provident fund = 0.15Q + 0.1Q = 0.25Q

Total annual contributions to provident fund = ₹ (0.25Q x 12) = ₹ 3Q.

If A be the total accumulation at the end of 30 year.

Then $A = \frac{P}{i}\left\{(1+i)^n - 1\right\}$ Here P = 3Q, $i = \frac{11}{100} = 0.11$, n = 30

$$\therefore\ A = \frac{3Q}{0.11}\left\{(1+0.11)^{30} - 1\right\} = \frac{3Q}{0.11}(22.89 - 1)$$

$$\therefore\ A = \frac{3Q}{0.11} \times 21.89 = 597Q$$

So for each ₹ 100 of the person's salary, the accumulation = ₹ (597× 100) [∵Q = 100]

= ₹ 59,700.

Let $x = (1.11)^{30}$

$\therefore \log x = \log (1.11)^{30}$

$= 30 \times \log 1.11$

$= 30 \times .0453$

$= 1.359$

$x = \text{Antilog } 1.359$

$= 22.89$

Example 30 :

A loan of ₹ 10,000 is to be repaid in 30 equal annual instalments of ₹ P. Find P if the compound interest charged is at the rate of 4% p.a.

Given $(1.04)^{30} = 3.2434$.

Solution :

Present Value = $V = \frac{P}{i}\left\{1 - (1+i)^{-n}\right\}$ Here V = ₹ 10,000

$10{,}000 = \frac{P}{0.04}\left\{1 - (1+0.04)^{-30}\right\}$ $i = \frac{4}{100} = 0.04$

or, $10{,}000 = \frac{P}{0.04}\left\{1 - (1.04)^{-30}\right\}$ n = 30

or, $10{,}000 \times 0.04 = P\ \ 1 - \frac{1}{3.2434}$ or, $400 = P \times \frac{2.2434}{3.2434}$

or, $P = \frac{400 \times 3.2434}{2.2434}$ = ₹ 578.30.

Example 31 :

A Professor retires at the age 60 years. He will get the pension of ₹ 42,000 a year paid in half-yearly instalment of rest of his life. Reckoning his expectation of life to be 15 years and that interest is at 10% p.a. payable half-yearly. What single sum is equivalent to his pension?

Solution :

The amount of pension = ₹ 42,000

$\therefore$ P = ₹ 42,000; n = 15 = number of years

i = 10% payable half-yearly = .10

We have,

$$V = \frac{P}{i}\left[1-(1+i)^{-2n}\right] = \frac{42,000}{0.10}\left[1-\left(1+\frac{0.10}{2}\right)^{-30}\right]$$

$= 4,20,000\ [1 - (1.05)^{-30}] = 4,20,000\ (1 - 0.23138)$

$= 4,20,000 \times 0.76862 = 3,22,820.40$

Hence, ₹ 3,22,820.40 is the amount of single sum equivalent to his pension.

Example 32 :

A man purchased a house valued at ₹ 3,00,000. He paid ₹ 2,00,000 at the time of purchase and agreed to pay the balance with interest of 12% per annum compounded half yearly in 20 equal half yearly instalments. If the first instalment is paid after six months from the date of purchase, find the amount of each instalment. [Given log 10.6 = 1.0253 and log 31.19 = 1.494]

Solution :

Since ₹ 2,00,000 has been paid at the time of purchase when cost of house was ₹ 3,00,000, we have to consider 20 equated half yearly annuity payment ₹ P when 12% is rate of annual interest compounded half yearly for present value of ₹ 1,00,000.

$\therefore 1,00,000 = \frac{2P}{0.12}\ [1 - (1 + .06)^{-20}]$ or, $1,00,000 \times 0.12 = 2P\ [1 - (1.06)^{-20}]$

or, 12,00,000 = 2P [1 – .3119] or, 6,00,000 = P x .6881

$\therefore P = \frac{6,00,000}{.6881}$ = ₹ 8,718.40.

$\therefore$ Amount of each instalment = ₹ 8,718.40.

Let $x = (1.06)^{-20}$

$\therefore \log x = -20 \log 1.06 = -20 \times .0253$

$= -0.506 = \bar{1}.496 = \log .3119$

$\therefore x = 0.3119$

Self Examination Questions

1. Mr. S Roy borrows ₹ 20,000 at 4% compound interest and agrees to pay both the principal and the interest in 10 equal annual instalments at the end of each year. Find the amount of these instalments. [*Ans.* ₹ 2,466.50]
2. A man borrows ₹ 1,000 on the understanding that it is to be paid back in 4 equal instalments at intervals of six months, the first payment to be made six months after the money was borrowed. Calculate the value of each instalment, if the money is worth 5% p.a. [*Ans.* ₹ 266]
3. A persons invests ₹ 1,000 every year with a company which pays interest at 10% p.a. He allows his deposits to accumulate with the company at compound rate. Find the amount standing to his credit one year after he has made his yearly investment for the tenth time? [*Ans.* ₹ 17,534]

4. A loan of ₹ 5,000 is to be paid in 6 equal annual payments, interest being at 8% per annum compound interest and the first payment be made after a year. Analyse the payments into those on account of interest and an account of amortisation of the principal.

[*Ans.* ₹ 1,081.67]

5. Mrs. S. Roy retires at the age of 60 and earns a pension of ₹ 60,000 a year. He wants to commute on-fourth of his pension to ready money. If the expectation of life at this age be 15 years, find the amount he will receive when money is worth 9% per annum compound. (It is assumed that pension for a year is due at the end of the year). [*Ans.* ₹ 1,20,910.55]

6. A Government constructed housing flat costs ₹ 1,36,000; 40% is to be paid at the time of possession and the balance reckoning compound interest @ 9% p.a. is to be paid in 12 equal annual instalments. Find the amount of each such instalment. [Given $\frac{1}{(1.09)^{12}} = 0.35587$]

7. Find the present value of an annuity of ₹ 300 p.a. for 5 years at 4%. Given log 104 = 2.0170333, log 0.0821923 = $\bar{2}.9148335$. [*Ans.* ₹ 1,335.58]

8. A person purchases a house worth ₹ 70,000 on a hire purchase scheme. At the time of gaining possession he has to pay 40% of the cost of the house and the rest amount is to be paid in 20 equal annual instalments. If the compound interest is reckoned at $7\frac{1}{2}$% p.a. What should be the value of each instalment? [*Ans.* ₹ 4,120]

1.3 DISCOUNTING OF BILLS AND AVERAGE DUE DATE

Few Definitions :

Present Value (P.V.) : Present value of a given sum due at the end of a given period is that sum which together with its interest of the given period equals to the given sum i.e.

P.V. + Int. on P.V. = sum due [Sum due is also known as Bill Value (B.V.)]

Symbols : If A = Sum due at the end of n years, P = Present value, i = int. of ₹ 1 for 1 yr. n= unexpired period in years, then A = P+P n i = P(1+n i).......(i)

or, $P = \frac{A}{1+ni}$

True Discount (T.D) :

True discount of a given sum due at the end of a given period, is the interest on the present value of the given sum i.e. T.D. = P n i..............(ii)

T.D.= Int. of P.V. = amount due – Present value i.e. T.D. = A – P.........(iii)

Again T.D. = $A - \frac{A}{1+ni} = \frac{Ani}{1+ni}$............(iv)

i.e. Find P.V. and T.D. of ₹ 327 due in 18 months hence at 6% S.I..

$$T.D = \frac{Ani}{1+ni} = \frac{327 \times \frac{3}{2} \times \frac{6}{100}}{1+\frac{3}{2} \times \frac{6}{100}} = 27,$$ here A = 327, n = 18 m = 3/2 yrs. i= 6/100.

We know P.V. +Int. on PV (i.e.T.D)= sum due (i.e.B.V)

Or, P.V. = B.V.– T.D. = 327 – 27 = ₹ 300.

SOLVED EXAMPLES :

(T.D; n; i are given, to find A)

Example 33 : The true discount on a bill due 6 months hence at 8% p.a. is ₹ 40, find the amount of the bill.

In the formula, $T.D = \frac{Ani}{1+ni}$, T.D. = 40, $n = \frac{6}{12} = \frac{1}{2}$, i = 8/100 = 0.08

$\therefore 40 = \frac{A.\frac{1}{2}.(0.08)}{1+\frac{1}{2}(0.08)}$, or, A = 1040 (in ₹)

Example 34 : (T.D.; A, i are given, to find n)

Find the time when the amount will be due if the discount on ₹ 1,060 be ₹ 60 at 6% p.a.

$T.D. = \frac{Ani}{1+ni}, or, 60 = \frac{1060.n.(0.06)}{1+n(0.06)}$ or, $n = \frac{2}{3}$ yrs.

Example 35 : (T.D.; A; n are given, to find i)

If the discount on ₹ 11,000 due 15 months hence is ₹ 1,000, find the rate of interest,

Here, A =11,000, $n = \frac{15}{12} = \frac{5}{4}$, T.D. =1000, i = ?

So we have, $1000 = \frac{11000 \times \frac{5}{4} \times i}{1+\frac{5}{4} \times i}$ or, 12500 i = 1000 or i = 0.08.

Example 36 : (If A, n; i; are given to find T.D.)

Find the T.D. on a sum of ₹ 1750 due in 18 months and 6% p.a.

Here $A = 1750; n = \frac{18}{17}$ yrs. $= \frac{3}{2}$.yrs; $i = \frac{6}{100} = 0.06$

So we get $T.D. = \frac{1750 \times \frac{3}{2} \times 0.06}{1+\frac{3}{2} \times 0.06} = \frac{1750 \times 0.09}{1+0.09}$ = ₹144.50 (approx)

$\therefore$reqd. T. D. = ₹ 144.50.

Example 37 : Find the present value of ₹ 1800 due in 73 days hence at 7.5% p.a. (take 1 year = 365 days)

Here : A = 1800, $n = \frac{73}{365} = \frac{1}{5}$ years; $i = \frac{7.5}{100} = 0.075$

$T.D. = \frac{1800.\frac{1}{5}.(0.075)}{1+\frac{1}{5}(0.075)} = \frac{1800 \times 0.015}{1+0.015} = \frac{27}{1.015}$ = ₹26.60

We know P.V. = A – T.D. $\therefore$ P.V. = 1800 – 26.60 = 1773.40

Example 38 : The difference between interest and true discount on a sum due in 5 years at 5% per annum is ₹ 50. Find the sum.

Let sum = ₹ 100, Interest = 100 × 5 × 0.05 = ₹ 25, $i = \frac{5}{100} = 0.05$

Again $T.D. = \frac{Ani}{1+ni} = \frac{100 \times 5 \times 0.05}{1+0.05} = \frac{25}{1.05} =$ ₹ 20; So difference = ₹(25 – 20) = ₹5

Diff	Sum
5	100
50	?

$= \frac{100}{5} \times 50 = 1000,$

Example 39 : If the interest on ₹ 800 is equal to the true discount on ₹ 848 at 4% When the later amount be due?

T.D. = A – P.V. = 848 – 800 = ₹ 48, here A = 848, P.V. 800

Again T.D. = P x n x i or, 48 = 800 × n × 0.04, or, $n = 1\frac{1}{2}$ yrs.

Self Examination Questions

1. The true discount on a bill due $1\frac{1}{2}$ years hence $4\frac{1}{2}$% p.a. is ₹ 54. Find the amt. of the bill? [Ans. ₹ 854]

2. The true discount on a bill due 146 days hence at $4\frac{1}{2}$%p.a. is ₹ 17. Find the amount of the bill (take 1 year = 365 days) [Ans. ₹ 961.44]

3. When the sum will be due if the present worth on ₹ 1662.25 at 6%p.a. amount to ₹ 1,525.[Ans. $1\frac{1}{2}$ yrs.]

4. Find the time that sum will be due if the true discount on ₹ 185.40 at 5% p.a. be ₹ 5.40 (taking 1 year = 365 days) [Ans. 219 days]

5. If the true discount on ₹ 1770 due $2\frac{1}{2}$ years hence, be ₹ 170, find the rate percent. [Ans. $1\frac{1}{2}$% p.a]

6. If the present value of a bill of ₹ 1495.62 due $1\frac{1}{4}$ years hence, be ₹ 1424.40; find the rate percent. [Ans. 4% p.a.]

7. Find the present value of ₹ 1265 due $2\frac{1}{2}$ years at 4% p.a. [Ans. ₹1150]

8. The difference between interest and true discount on a sum due 73 days at 5% p.a. is Re.1. Find the sum. [Ans. ₹ 10,000]

9. The difference between interest and true discount on a sum due $2\frac{1}{2}$ years at 4% p.a. is ₹ 18.20. Find the sum. [Ans. ₹ 20,000]

10. If the interest on ₹ 1200 in equal to the true discount on ₹1254 at 6%, when will the later amount be due ? [Ans. 9 months]

1.3.1 BILL OF EXCHANGE :

This is a written undertaking (or document) by the debtor to a creditor for paying certain sum of money on a specified future date.

A bill thus contains (i) the drawer (ii) the drawee (iii) the payee. A specimen of bill is as follows

Stamp	Address of drawer Date......
Six months after date pay to M/s. E.P.C. or order the sum of ₹ 1000 (rupees one thousand only) for the value received.	
C.K. Basu 1/1 S.K. Dhar Rd. Kolkata – 700 017	A. B. Chakraborty (drawer)

Bill of Exchange is two kinds

(i) Bill of exchange after date, in which the date of maturity is counted from the date of drawing the bill.

(ii) Bill of exchange after sight, in which the date of maturity is counted from the date of accepting the bill.

The date on which a bill becomes due is called nominal due date. If now three days, added with this nominal due date, the bill becomes legally due. Thus three days are known as days of grace.

Banker's Discount (B.D.) & Banker's Gain (B.G.):

Banker's discount (B.D.) is the interest on B.V. and difference between B.D. and T.D. is B.G.

i.e. B.D. = int. on B.V. = Ani ....(v)

B.G. = B.D. – T.D. and B.G. = interest on T.D.

$$B.G. = \frac{A(ni)^2}{1+ni} \text{.............(vi)}$$

B.V. – B.D. = Discounted value of the bill......(vii)

SOLVED EXAMPLES :

Example 40 : A bill for ₹ 1224 is due in 6 months. Find the difference between true discount and banker's discount, the rate of interest being 4% p.a.

$$T.D = \frac{Ani}{1+ni} = \frac{1224 \times \frac{1}{2} \times (0.04)}{1+\frac{1}{2} \times (0.04)} = \frac{24.48}{1.02} = 24.$$ B.D. = Ani = 1224 x $\frac{1}{2}$ x (0.04) = 24.48;

B.D. – T.D. = 24.48 – 24= 0.08; ∴ required difference = ₹ 0.48 [This difference is B.G. (0.48)

Again Int. on 24 (i.e., T.D.) $= 24.\frac{1}{2}.(0.04) = 0.48$, i.e, B.G. = Int. on T.D.]

Example 41 : If the difference between T.D. and B.D. on a sum due in 4 months at 3% p.a. is ₹ 10, find the amount of the bills.

B.G. = B.D. – T.D. = 10; $n = 4/12 = \frac{1}{3}$ yrs.; $i = \frac{3}{100} = 0.03$; A = ?

B.D. = A ni, T.D. = Pni, B.G. = B.D. – T.D.= A ni – Pni = (A –P) ni

Now, (A – P) ni = 10or (A – P) $\frac{1}{3}$. (0.03) = 10,

or, (A–P) $= \frac{10}{0.01} = 100$

or, T.D. = 1000 (as T.D. = A – P.) Again T.D. = P ni,

or, $P.\frac{1}{2}$ (0.03) = 1000 or, $P = \frac{1000}{0.01}$

∴ P = 1,00, 000. Now, A = P.V. + T.D. = 1,00,000 + 1000 = 1,01,000

∴ required amount of the bill = ₹ 1,01,000.

Aliter : $B.G. = \frac{A(ni)^2}{1+ni}$, $10 = \frac{A.\left(\frac{1}{3}\times.03\right)^2}{1+\frac{1}{3}\times.03}$, A = 1,01,000.

Example 42 : The T.D. and B.G. on a certain bill of exchange due after a certain time is respectively ₹ 50 and Re. 0.50. Find the face value of the bill.

We know B.G. = Int. on T.D. or, 0,50 = 50 × n × i or, $ni = \frac{0.50}{50} = 0.01$

Now, B.D. = T.D + B.G. or, B.D. = 50 + 0.50 = 50.50

Again B.D. int. on B.V. (i.e., A)

or, 50.50 = A. ni or, 50.50 = A.. (0.01) or, $A = \frac{50.50}{0.01} = 5050$

∴ reqd. face value of the bill is ₹ 5050.

Example 43 : A bill of exchange drawn on 5.1.2013 for ₹ 2,000 payable at 3 months was accepted on the same date and discounted on 14.1.13, at 4% p.a. Find out amount of discount.

Unexpired number of days from 14 Jan to 8 April = 17 (J) + 28 (F) + 31 (M) + 8 (A) = 84 (excluding 14.1.13)

2013 is not a leap year, Feb. is of 28 days. $B.D. = 2000 \times \frac{84}{365} \times \frac{4}{100} = 18.41$ (after reduction)

Hence, reqd. discount = ₹ 18.41.

Drawn on period	5.1.13
Period	3
Nominally due on	5.4.13
Days of grace	3
Legally due	8.4.13

SELF EXAMINATION QUESTIONS

Problems regarding T.D. B.D. B.G.

1. At the rate of 4% p.a. find the B.D., T.D. and B.G. on a bill of exchange for ₹ 650 due 4 months hence. [Ans. ₹ 8.67; ₹ 8.55; Re. 0.12]
2. Find the difference between T.D. and B.D. on ₹ 2020 for 3 months at 4% p.a. Show that the difference is equal to the interest on the T.D. for three months at 4%. [Ans. Re. 0.20]
3. Find the T.D. and B.D. on a bill of ₹ 6100 due 6 months hence, at 4% p.a. [Ans. ₹ 119.61; ₹122]
4. Find out the T.D. on a bill for 2550 due in 4 months at 6% p.a. Show also the banker's gain in this case. [Ans. ₹ 50, Re.1]
5. Find the T.D. and B.G. on a bill for ₹ 1550 due 3 months hence at 6% p.a. [Ans. ₹ 22.9]; ₹ 0.34]
6. Calculate the B.G. on ₹ 2500 due in 6 months at 5% p.a. [Ans. ₹ 1.54]
7. If the difference between T.D. and B.D. on a bill to mature 2 months after date be Re. 0.25 at 3% p.a.; find. (i) T.D. (ii) B.D. (iii) amount of the bill. [Ans. ₹ 50; ₹ 50.25; ₹10,050]
8. If the difference between T.D. and B.D.(i.e. B.G.) on a sum of due in 6 months at 4% is ₹ 100 find the amount of the bill. [Ans. ₹ 2,55,000]

   [hints : refer solved problem no 55]
9. If the difference between T.D. and B.D. of a bill due legally after 73 days at 5% p.a. is ₹10, find the amount of the bill. [Ans. 1,01,000]
10. If the banker's gain on a bill due in four months at the rate of 6% p.a. be ₹ 200, find the bill value, B.D. and T.D. of the bill.

    [Ans.5,10,000; ₹ 10,200; ₹10,000]
11. A bill for ₹ 750 was drawn on 6th March payable at 6 months after date, the rate of discount being 4.5% p.a. It was discounted on 28th June. What did the banker pay to the holder of the bill?

    [Ans. ₹ 743.62]
12. A bill of exchange for ₹ 846.50 at 4 months after sight was drawn on 12.1.2013 and accepted on 16th January and discounted at 3.5% on 8th Feb.2013. Find the B.D. and the discounted value of the bill. [Ans. ₹ 8.18; ₹ 838.32]
13. A bill of exchange for ₹ 12,500 was payable 120 days after sight. The bill was accepted on 2nd Feb.2013 and was discounted on 20th Feb. 2013 at 4%. Find the discounted value of the bill.

    [Ans. 12,356.17]
14. A bill for ₹ 3,225 was drawn on 3rd Feb. at 6 months and discounted on 13th March at 8% p.a. For what sum was the bill discounted and how much did the banker gain in this?

    [Ans. ₹ 3121.80; ₹ 3.20]

Problems regarding rate of interest :

15. What is the actual rate of interest which a banker gets for the money when he discounts a bill legally due in 6 months at 4% p.a. [Ans. 4.08% approx]
16. What is the actual rate of interest which a banker gets for the money when he discounts a bill legally due in 6 months at 5%. [Ans. $5\frac{3}{9}$%]
17. If the true discounted of a bill of ₹ 2613.75 due in 5 months be ₹ 63.75; find rate of interest.

    [Ans. 6%]

1.3.2 AVERAGE DUE DATE

Meaning

Average Due date is the mean or equated date on which a single payment of an aggregate sum may be made in lieu of several payments due on different dates without, however, involving either party to suffer any loss of interest, i.e, the date on which the settlement takes place between the parties is known as Average or Mean Due Date.

This is particularly helpful in the settlement of the following types of accounts, viz., :

(i) in case of accounts which are to be settled by a series of bills due on different dates ; (ii) in case of calculation of interest on drawings of partners; (iii) in case of piecemeal distribution of assets during partnership dissolution etc.; and (iv) in the case of settlement of accounts between a principal and an agent.

Types of Problems

Two types of problems may arise. They are :

(1) Where amount is lent in **various** instalments but repayment is made in **one** instalment only;

(2) Where amount is lent in **one** instalment but repayment is made in **various** instalments.

Method (1) : Where amount is lent in various instalments but repayment is made in one instalment :

Step 1. *Take up the starting date'(preferably the earliest due date as '0' date or 'base date' or 'starting date');*

Step 2. *Calculate the number of days from '0' date to each of the remaining due dates;*

Step 3. *Multiply each amount by the respective number of days so calculated in order to get the product;*

Step 4. *Add up the total products separately;*

Step 5. *Divide the total products by the total amounts of the bills;*

Step 6. *Add up the number of days so calculated with '0' date in order to find out the Average or Mean Due Date.*

Date of Maturity and Calculations

If there is an *after date bill,* the period is to be counted from the *date of drawing the bill* but when there is any *after sight bill,* the said period is to be counted from the *date of acceptance of the bill.* For example, if a bill is drawn on 28th January 2013, and is made payble at one month after date, the due date will be 3rd day after 28th Feb. i.e., **2nd** March 2013.

To Sum up

(i) *When the period of the bill is stated in days, the date of maturity will also be calculated in terms of days i.e., excluding the date of transaction but including the date of payment.*

E.g. If a bill is drawn on 18th January 2012 for 60 days, the maturity will be 21st March 2012.

(ii) *If the period of the bill is stated in month, the date of maturity will also be calculated in terms of month neglecting, however, the number of days in a month.*

E.g. If a bill is drawn on 20th May, 2012 for 3 months, of date of maturity will, naturally, be 23rd August, 2012.

(iii) *What the date of maturity of a bill falls on 'emergent holiday' declared by the Government, the date of maturity will be the next working day.*

(iv) *When the date of maturity of a bill falls on a public holiday, the bill shall become due on the next preceding business day and if the next preceding day again falls on a public holiday, it will become due on the day preceding the previous day — Sec. 25.*

E.g. If the date of maturity of a bill falls on 15th August (Independence day) it falls due on 14th August. But if 14th August falls again on a public holiday, the 13th August will be . considered as the date of maturity.

Example 44 :

Calculate Average Due date from the following information:

Date of the Bill	Term	Amount (₹)
August 10, 2011	3 months	6,000
October 23, 2011	60 days	5,000
December 4, 2011	2 months	4,000
January 14, 2012	60 days	2,000
March 8, 2012	2 months	3,000

Solution :

Computation of Average Due Date **'O' Date = 13.11.1994**

Date of the Bill	Due Date	No. of days from '0 date	Amount	Product
			₹	₹
10.8.2011	13.11.2011	0	6,000	0
23.10.2011	25.12.2011	42(17 + 25)	5,000	2,10,000
4.12.2011	7.2.2012	86 (17 + 31+31+7)	4,000	3,44,000
14.1.2012	18.3.2012	125 (17 + 31 + 31 + 28 +18)	2,000	2,50,000
8.3.2012	11.5.2012	179(17 + 31+31+28 + 31+30 + 11)	3,000	5,37,000
			20,000	13,41,000

$\therefore$ Average Number of Days = $\frac{₹ 13,41,000}{₹ 20,000}$ = 67 days (approx)

$\therefore$ So, Average Due Date will be = 13th Nov. 2011 + 67 days = 19.01.2012.

Working

Due date to be calculated as under

Date of the Bill	Periods	Due Date (after adding 3 days for grace)
10.8.2011	3 months	13.11.2011
23.10.2011	60 days	25.12.2011 (8 + 30 + 22 + 3)
4.12.2011	2 months	7.2.2012
14.1.2012	60 days	18.3.2012(17 + 28 + 15 + 3)
8.3.2012	2 months	11.5.2012

Example 45 :

Sardar sold goods to Teri as under:

Date of Invoice	Value of Goods Sold	Date of Invoice	Value of Goods Sold
	₹		₹
7.5.2012	1,000	24.5.2012	1,500
15.5.2012	2,000	1.6.2012	4,000
18.5.2012	3,500	7.6.2012	3,000

The payments were agreed to be made by bill payable 2 months (60 days) from the date of invoice. However, Teri wanted to make all the payments on a single date.

Calculate the date on which such a payment could be made without loss of interest to either party.

Solution :

Calculation of Average Due Date '0' Date = 06.07.2012

Date of Invoice	Due Date	No. of days from '0' Date	Amount ₹	Product ₹
7.5.2012	6.7.2012	0	1,000	0
15.5.2012	14.7.2012	8	2,000	16,000
18.5.2012	17.7.2012	11	3,500	38,500
24.5.2012	23.7.2012	17	1,500	25,500
1.6.2012	31.7.2012	25	4,000	1,00,000
7.6.2012	6.8.2012	31	3,000	93,000
			15,000	2,73,000

Average Number of Days $= \frac{₹\ 2,73,000}{₹\ 15,000}$ = 18 days (approx.)

∴ Average Due Date will be = 18 days from '0' date i.e., 6th July + 18 days = 24th July, 2012

Workings

Due dates to be calculated as under

Date of Invoice	Period in Days	Due Date			May +	June +	July +	Aug.		
7.5.2012	60	6.7.2012	(From 7.5.2012 + 60 days)	=	24 +	30 +	6		=	60 days
15.5.2012	60	14.7.2012	(From 15.5.2012 + 60 days)	=	16 +	30 +	14		=	60 days
18.5.2012	60	17.7.2012	(From 18.5.2012 + 60 days)	=	13 +	30 +	17		=	60 days
24.5.2012	60	23.7.2012	(From 24.5.2012 + 60 days)	=	7 +	30 +	23		=	60 days
1.6.2012	60	31.7.2012	(From 1.6. 2012 + 60 days)	=	—	29 +	31		=	60days
7.6.2012	60	6.8.2012	(From 7.6.2012 + 60 days)	=	—	23 +	31+6		=	60 days

Example 46 :

Ramkumar having accepted the following bills drawn by his creditor Prakash Chand, due on different dates, approached his creditor to cancel them all and allow him to accept a single bill for the payment of his entire liability on the average due date.

You are requested to ascertain the total amount of the bill and its due date.

Bill No.	*Date of Drawing*	*Date of Acceptance*	*Amount of the Bill* ₹		*Tenure*
(i)	16.2.2012	20.2.2012	8,000	90 days	after sight
(ii)	6.3.2012	6.3.2012	6,000	2 months	after sight
(iii)	24.5.2012	31.5.2012	2,000	4 months	after sight
(iv)	1.6.2012	1.6.2012	9,000	1 month	after sight

Solution : **'0' Date = 9.5.2012**

Computation of Average Due Date

Date of Drawing	Due Date	No. of days from '0' Date		Amount ₹	Product ₹
16.2.2012	24.5.2012	15		8,000	1,20,000
6.3.2012	9.5.2012	0		6,000	*'0'*
24.5.2012	3.9.2012	117	(22+30 + 31+31+3)	2,000	2,34,000
1.6..2012	4.7.2012	56	(22+30 + 4)	9,000	5,04,000
				25,000	8,58,000

Average Number of Days $= \dfrac{₹\ 8,58,000}{₹\ 25,000}$ = 34 days (approx)

∴ Average due date will be = 9th May + 34 days = 12th June.

Total amount of the bill = ₹ 25,000

Workings

Since all the bill are '*After Sight*' the period is to be computed from the *date of acceptance* of the bill.

Due Date to be calculated as under

Date of Acceptance	**Periods**	**Due Date (with days of grace)**		Feb + March + April + May
20.2.2012	90 days	24.5.2012	(From 20.2.2012+ 90days)	= 8 + 31 +30+21= 90days
6.3.2012	2 months	9.5.2012		
31.5.2012	4 months	3.9.2012		
1.6.2012	1 month	4.7.2012		

Example 47 :

For goods sold, Nair draws the following bills on Ray who accepts the same as per terms :

Amount of the Bill ₹	*Date of Drawing*	*Date of Acceptance*	*Tenure*	
8,000	6.1.2012	9.1.2012	3	months after date
9,000	15.2.2012	18.2.2012	60	days after date
8,000	21.2.2012	21.2.2012	2	months
15,000	14.3.2012	17.3.2012	30	days after sight

On 18th March 2011 it is agreed that the above bills will be withdrawn and the acceptor will pay the whole amount in one lump-sum by a cheque 15 days ahead of average due date and for this a rebate of ₹ 1,000 will be allowed. Calculate the average due date, the amount and the due date of the cheque.

Solution :

Computation of Averge Due Date **'0' Date = 9.4.2012**

Date of Drawing	*Due Date*	*No. of days from'0'date*	*Amount* ₹	*Product* ₹
6.1.2012	9.4.2012	0	8,000	0
15.2.2012	19.4.2012	10	9,000	90,000
21.2.2012	24.4.2012	15	8,000	1,20,000
14.3.2012	19.4.2012	10	15,000	1,50,000
			40,000	3,60,000

$$\text{Average number of days} = \frac{₹\ 3{,}60{,}000}{₹\ 40{,}000} = 9 \text{ days}$$

∴ Average Due Date = April 9 + 9 days = April 18th 2012.

Amount of cheque = ₹ 40,000 - ₹ 1,000 - ₹ 39,000

Due Date of the cheque = April 18th -15 days = April 3, 2012

Workings

Due Dates to be calculated as under

It must be remember that in case of **After** Date bill, date of maturity is to be calculated from the **date** of drawing of the bill but in case of **After Sight bill,** the date of maturity is to be calculated from the date of acceptance of the bill.

Date of Drawing	**Date of Acceptance**	**Period**	**Due Date**
6.1.2012	9.1.2012	3 months	9.4.2012
15.2.2012	18.2.2012	60 days	19.4.2012
21.2.2012	21.2.2012	2 months	24.4.2012
14.3.2012	17.3.2012	30 days	19.4.2012

Reciprocal Bills

Sometimes the two parties in a bill of exchange mutually agree to accept the reciprocal bills (i.e., each party draws and accepts, respectively). Under the circumstances, the Average Due Date Method car better be applied where there are different bills and different due dates. The method of calculation is sami as above. But after calculating the respective Average Due Date (for both Bills Receivable and Bill: Payable), find out the difference both in **amounts** and in **product** and average number of days may b< ascertained by dividing the former by the latter. Then add up the days so calculated with the '0' date ii order to find out the Average Due Date.

Consider the following illustration

Example 48 :

A.N.K. had the following bills receivable and bills payable against A.N.R.

Calculate the average due date when the payment can be made or received without any loss of interest to either party.

Notes: Holidays intervening in the period : 15 Aug., 16 Aug., 6th Sept. 2012.

	Bills Receivable			*Bills Payable*	
Date	*Amount* ₹	*Tenure (Months)*	*Date*	*Amount* ₹	*Tenure (Months)*
1.6.2012	3,000	3	29.5.2012	2,000	2
5.6.2012	2,500	3	3.6.2012	3,000	3
9.6.2012	6,000	1	10.6.2012	6,000	2
12.6.2012	10,000	2	13.6.2012	9,000	2
20.6.2012	15,000	3	27.6.2012	13,000	1

Solution :

Calculation of Average Due Date

The date of maturity of the Bills Receivable (after adding 3 days as grace) by A.N.R. are : 4. 9. 2012; 8.9.2012; 12.7.2012; 14.8.2012 and 23.9.2012. And the date of maturity of the Bills Payable by A.N.R. are : 1.8.2012; 5.9.2012; 13.8.2012; 14.8.2012 and 30.7.2012. 12th July 2009 is taken as the '0' date.

(i) Bills Receivable against A.N.R. : **'0' Date = 12.7.2012**

Computation of Average Due Date

Due Date		*No. of days from '0' date*	*Amount* ₹	*Product* ₹
(1)		(2)	(3)	(4)
		(J + A + S)	3,000	1,62,000
4.9.2012	54	(19 + 31+4)		
8.9.2012	58	(19 + 31+8)	2,500	1,45,000
12.7.2012	0		6,000	0
14.8.2012	33	(19 + 14)	10,000	3,30,000
23.9.2012	73	(19 + 31 + 23)	15,000	10,95,000
			36,500	17,32,000

(ii) Bills Payable against A.N.R. : **'0' Date = 12.7.2012**

1.8.2012	20	(19 + 1)	2,000	40,000
5.9.2012	55	(19 + 31+5)	3,000	1,65,000
13.8.2012	32	(19 + 13)	6,000	1,92,000
14.8.2012	33	(19 + 14)	9,000	2,97,000
30.7.2012	18	(18)	13,000	2,34,000
			33,00	9,28,000

Now adjustment for the Bill :		
Total Bills Receivable against A.N.R.	36,500	17,32,000
Total Bills Payable against A.N.R.	33,000	- 9,28,000
	3,500	8,04,000

$$\text{Average number of days} = \frac{₹\ 8,04,000}{₹\ 3,500} = 230 \text{ days (approx.)}$$

∴ Average due date will be = July 12,2009 + 230 days = Feb. 27,2010.

Note : If the date of maturity of a bill falls due on a public holiday the maturity date will be just the next preceding business day, e.g., bill which matures on Aug. 15 becomes due on Aug. 14.

Calculation of Interest

The Average Due Date method of computation, practically, simplifies interest calculations. The interest may be calculated from the Average Due Date to the date of settlement of accounts instead of making separate calculation for each bill. The interest so calculated will neither affect the creditors nor the debtors, i.e., both the debtors and the creditors will not lose interest for the period.

Under the circumstances, the creditors/drawers will receive and debtors/drawees will pay the interest at the specified rate for the delayed period. This is particularly applicable while calculating interest on partners' drawing in case of partnership firms.

Method of Calculation

Step 1. *At first calculate the Average Due Date in the usual way.*

Step 2. *Ascertain the difference between the so ascertained Average Due Date and the date of closing the books.*

Step 3. *Interest is calculated with the help of the following:*

$$\textit{Interest} = \text{Amount} \times = \frac{\text{Rate of interest}}{100} \times \frac{\text{No. of months}}{12}$$

Consider the following illustration :

Example 49 :

Satyajit and Prosenjit are two partners of a firm. They have drawn the following amounts from the firm ir the year ending 31st March 2012:

	Satyajit			*Prosenjit*	
		₹			₹
2011:	Date		2011:	Date	
	1st July	300		1st June	500
	30th Sept.	500		1st August	400
	1st Nov.	800			
2012:	2012:				
	28th February	200		1st February	400
				1st March	900

Interest at 6% is charged on all drawings.

Calculate interest chargeable under Average Due Date System. (Calculation to be made in months.)

Solution :

(a) Calculation of Interest in case of Satyajit:

Computation of Average Due Date **'0' Date = 1.7.2011**

Date	No. of days from '0' date J + A + S		Amount ₹	Product ₹
1.7.2011	0		300	0
30.9.2011	91	(30 + 31 + 30 + 1)	500	45,500
1.11.2011	123	(30 + 31 + 30 + 31+1)	800	98,400
28.2.2012	242	(30 + 31 + 30 + 31+30 + 31+31+28)	200	48,400
			1,800	1,92,300

$\therefore$ Average number of days = $\frac{₹ 1,92,300}{₹ 1,800}$ = 107 days (approx.)

Average due date = July 1 + 107 days = 16th Oct. 2011.

Therefore, interest is chargeable from 16.10.2011 to 31.3.2012, i.e., $5\frac{1}{2}$ months.

$$= ₹\ 1,800 \times \frac{6}{100} \times \frac{5\frac{1}{2}}{12} = ₹\ 49.50$$

(b) Calculation of Interest in case of Prosenjit:

Computation of Average Due Date **'0' Date = 1.6.2012**

Date		*No. of days from '0' date*	*Amount* ₹	*Product* ₹
1.62011	0		500	0
1.8.2011	61	(29 + 31 + 1)	400	24,400
1.2.2012	245	(29 + 31+31+30 + 31+30 + 31+31 + 1)	400	98,000
1.3.2012	273	(29 + 31+31+30 + 31+30 + 31+31+28 + 1)	900	2,45,700
			2,200	3,68,100

$\therefore$ Average number of days = $\frac{₹ 3,68,100}{₹ 2,200}$ = 167 days (approx.)

Average Due Date will be June 1 + 167 days = 16.11.2011.

$\therefore$ Interest is chargeable from 16.11.2011 to 31.3.2012 = $4\frac{1}{2}$ months

Amount of interest will be = $₹\ 2,200 \times \frac{6}{100} \times \frac{4\frac{1}{2}}{12} = ₹ 49.50$.

In case of Mutual dealings between the parties

It frequently happens in real-world situation that an individual plays both the role of a debtor as well as a creditor, i.e., mutual dealings. For example. Mr. X purchases raw materials from Y and he sells finished products to him. Consequently, for settlement purpose, who owes the higher amount pays the differece to the other person together with interest for the balance.

The following steps should carefully be followed for the purpose :

Step I *Find out the Product of both receivable and payable in usual manner like ordinary average due date method.*

Step II *Find out the difference of the product.*

Step III *Find out the difference of the amounts.*

Step VI *Average Due Date* = '0' + $\frac{\text{Difference in Product}}{\text{Difference in Amount}}$

Step V *For Calculation of Interest:*

$$= \text{Balance of Amount} \times \frac{\text{Rate of Interest}}{100} \times \frac{\text{Settlement Date - Average Due Date}}{\text{365 days or 12 months}}$$

Example 50 :

Mr. Big purchased goods from Mr. Small as follows :

₹ 4,000 to be paid on 6th January 2012

₹ 2,000 to be paid on 3rd February 2012

₹ 3,000 to be paid on 31st March 2012

He also sold goods to Mr. Small as follows:

₹ 1,500 to be paid on 10th January 2012

₹ 2,500 to be paid on 15th February 2012

₹ 1,000 to be paid on 21st March 2012

Mr. Big settled the account on 21st April 2012

(a) Find out the average due date.

(b) Calculate the interest at 5% p.a. from the average due date to the date of settlement.

Solution :

Calculation of Average Due Date

(a) *For Big's Purchases*

6th January 2012 is taken as the starting date or '0' date

Date of Transaction	Amount ₹	No. of days from '0'date	Product ₹
6.1.2012	4,000	0	0
3.2.2012	2,000	28	56,000
31.3.2012	3,000	84	2,52,000
	9,000		3,08,000

(b) *For Big's Sales*

Date of Transaction	Amount ₹	No. of days from '0'date	Product ₹
10.1.2012	1,500	40	6,000
15.2.2012	2,500	40	1,00,000
21.3.2012	1,000	74	74,000
	5,000		1,80,000

Now, the difference in Product is ₹ 1,28,000 (i.e., ₹ 3,08,000 - ₹ 1,80,000) and the difference in amount is ₹4,000 (i.e., ₹ 9,000 - ₹ 5,000).

No. of days from '0' date = $\frac{₹\ 1,28,000}{₹\ 4,000}$ = 32 days

i.e., Average Due Date = 6.1.2012 + 32 days = 7.2.2012

Date if Settlement = 21.4.2012

Hence, No. of days from average due date to the date of settlement = 73 days

$\therefore$ Amount of interest will be ₹ 40 i.e. $₹\ 4,000 \times \frac{73}{365} \times \frac{5}{100}$

Method (2) Where amount is lent in one instalment but repayment is made by various instalments This method is just the opposite of Method (1) Stated above

Method of Calculation

Step 1. Calculate the number of days from the date of lending to the date of each repayment made.

Step 2. Ascertain the sum of those days/months/years.

Sept 3. Divide the sum so ascertained (in Step 2) by the number of instalments paid.

Step 4. The result will be the number of days/months by which the Average Due Date falls from the date of taking such loans. .

The same will be calculated with the help of the following :

Average Due Date:

= Date of taking loan +

$$\frac{\text{Sum of days / months / years from the date of lending to date of repayment of each instalment}}{\text{No. of instalments}}$$

Consider the following illustration :

Example 51 :

When repayment of loan is made by monthly instalments

X lent ₹ 3,000 to Y on 1st January 2012 to be repaid by 5 equal monthly instalments starting fron 1st February subject to interest at 10% p.a. Y intends to repay the loan by single payment on average du< date. Find out that date and total interest payable.

Solution :

Average Due Date = Date of taking loan +

$$\frac{\text{Sum of days/months/years from the date of lending to the date of repayment of each instalment}}{\text{No. of instalments}}$$

$= \text{Jan 1.} + \frac{1+2+3+4+5}{5}$

= Jan. 1+3 months

= April 1.

Alternatively, the same can also be calculated as : '0' Date = 1.1.2012

Computation of Average Due Date

Starting date	*Due Date*	*Amount* ₹	*No. of months from Jan. 1*	*Product*
(1)	(2)	(3)	(4)	(5)
Jan. 1 2012	Feb. 1 2012	600	1	600
Jan. 1 2012	March 1 2012	600	2	1,200
Jan. 1 2012	April 1 2012	600	3	1,800
Jan. 1 2012	May 1 2012	600	4	2,400
Jan. 1 2012	June 1 2012	600	5	3,000
		3,000		9,000

$$\text{Average} = \frac{₹\ 9,000}{₹\ 3,000} = 3 \text{ months}$$

∴ Average due date is 3 months from Jan. 1. i.e., Jan 1 + 3 months = April 1.
Interest chargeable @ 10% on ₹ 3,000 for 3 months (April 1 to June 1) should be calculated

$$\text{That is, interest will be} = ₹\ 3,000 \times \frac{10}{100} \times \frac{3}{12} = ₹\ 75.$$

Example 52 :

X accepted the following bills drawn by his creditor Y :

Feb. 10 for ₹ 4,000 at 3 months.
March 15 for ₹ 5,000 at 3 months.
April 12 for ₹ 6,000 at 3 months.
May 8 for ₹ 5,000 at 5 months.

On 1st June it was decided that X would withdraw all such bills and he should accept on that day two bills, one for ₹ 12,000 due in 3 months and the other for the balance for 5 months.

Calculate the amount of the second bill after taking into consideration that rate of interest is @ 10% p.a. Ignore days of grace.

Solution:

Take May 10 as '0' date; the following table is prepared on that basis :

Compution of Average Due Date **'0' Date = 1.5.2012**

Date of Drawing		*Periods*	*Due Date of the original bills*		*Amount* ₹	*No. of days from'0'date*		*Product* ₹
Feb.	10	3	May	10	4,000	0		0
March	15	3	June	15	5,000	36	(21 + 15)	1,80,000
April	12	3	July	12	6,000	33	(21 + 12)	1,98,000
May	8	5	Oct.	8	5,000	151	(21+30 + 31+31+30 + 8)	7,55,000
					20,000			11,33,000

∴ Average number of days = ₹ 11,33,000/₹ 20,000 = 57 days (approx.)

The average due date will be May 10+57 days, i.e., July 6.

Due date of the new bills = Sept. 1 and Nov. 1 .

Calculation for interest @ 10% p.a. on—

				₹
₹	12,000	from July 6 to Sept. 1 i.e., for 57 days = ₹ 12,000 $\times \frac{10}{100} \times \frac{57}{365}$	=	187
₹	8,000	from July 6 to Nov. 1. i.e., for 118 days = ₹ 8,000 $\times \frac{10}{100} \times \frac{118}{365}$	=	259
	20,000			446

Therefore, amount of second bill will be ₹ 8,000 for the principal + ₹ 446 for interest = ₹ 8,446.

1.4 MATHEMATICAL REASONING – BASIC APPLICATION

The power of reasoning makes one person superior to the other. There are two types of reasoning : (i) Inductive reasoning (ii) Deductive reasoning.

The inductive reasoning is bassed on the principle of Mathematical Induction.

Here we shall discuss about deductive Reasoning. For this first we have to know about Mathematical statement or logical statements.

Mathematical Statement:

In our daily life we use different types of sentences like Assertive, Interrogative, Exclamatory, Imperative, Optative etc. Among them only assertive sentences are called Mathematical statement. But it is to be noted that all assertive sentences are not Mathematical Statements.

For example: 'The earth moves round the sun'. – This is a Mathematical statement. It is true always.

'The sun rises in the west'. – This is also a Mathematical statement. But its truth value is 'False'.

Again we take the example of assertive sentence:

'Girls are more clever than boys' – This is an assertive sentence but we can not say whether this sentence is always true or false. For this reasons this sentence is not a mathematical Statement.

Hence, we give the following definition of Mathematical Statement.

A sentence is called a mathematically acceptable statement or simply mathematical statement if it is true or false but not both.

Example 53 : The followings are the examples of Mathematical statements.

(i) 2 + 3 = 5. (ii) 3 + 4 = 6. (iii) Calcutta is the capital of West Bengal. (iv) Patna is in Orrisa.

(v) $\sqrt{5}$ is a rational number.

Example 54 : The followings are not the Mathematical Statements:

(i) $X^2 - 3X + 2 = 0$. (ii) Open the door. (iii) Give me a Pen.

(iv) Go to the market. (v) What do you want? (vi) How beautiful is the building!

Negation of a Statement:

Negation of a statement implies the denial or contradiction of the statement. If 'p' be a statement, then '~p' denotes the Negation of the statement.

For example: 'Manas is a teacher'. – It is a mathematical statement. Its negation is 'Manas is not a teacher' or It is false that 'Manas is a teacher'.

Again 'Delhi is the capital of India' is a mathematical statement.

Its negation statement is 'Delhi is not the capital of India' or, 'It is false that Delhi is the capital of India'.

Simple and compound Statements :

If the truth value of a statement does not depend on any other statement, then the statement is called a **simple Mathematical statement**. Simple statement cannot be subdivided into two or more **simple statements**.

A compound statement is a combination of two or more simple statements connected by the words "and", "or", etc.

A compound statement can be subdivided into two or more simple statements.

Example 55 : The followings are simple statements and they cannot be sub-divided into simpler statements.

(i) The earth moves round the sun.

(ii) The Sun is a star.

(iii) Sweta reads in class X.

(iv) 30 is a multiple of 5

(v) An integer is a rational number

Example 56 : The followings are compound statements and each of them can be sub-divided into two or more simple statements.

(i) 2 is a rational number and $\sqrt{2}$ is an irrational number.

(ii) A rhombus is a parallelogram and its four sides are equal.

(iii) A student who has taken Mathematics or computer Science can go for MCA.

(iv) The opposite sides of a parallelogram are parallel and equal.

Connectives:

Some connecting words are used to form compound statements. These connecting words are called ***connectives***. The connectives are the words namely: "and", "or", "if-then", "only if", "if and only if".

The word "and"

Any two simple statements can be combined by using the word "and" to form compound statements which may be true or false.

If each simple mathematical statements belonging to a compound mathematical statements are true then the compound mathematical statement is only ***True***. But if one or more simple statements connected with a compound mathematical statement is are false, then the compound mathematical statement must be ***<u>False</u>***.

Example 57 : (i) r: Calcutta is a big city and it is the capital of West Bengal.

The statement r is a compound mathematical statement and is formed by connecting two simple mathematical statements p & q using the connective "and" where

p: Calcutta is a big city.

q: Calcutta is the capital of West Bengal.

Here both p and q are true, so the truth value of the compound mathematical statement is **"*<u>True</u>*".**

(ii) r : 41 is a prime number and it is an even number

Here r is a compound mathematical statement and is formed by connecting two simple mathematical statements p and q using the connective "and" where

p : 41 is a prime number.

q : 41 is an even number.

Here p is true but q is false. So the truth value of r is **"*<u>False</u>*".**

Remarks :

All mathematical statements connected with "and" may not be a compound mathematical statement. For example, "The sum of 5 and 7 is 12". – It is a simple mathematical statement but not a compound mathematical statement.

The word "or":

Any two simple mathematical statements can be combined by using "**or**" to form a compound mathematical statements whose truth value may be true or false.

If both or any one of the component simple mathematical statements of a compound mathematical statements when formed using the connective "or" are / is true then the truth value of the compound mathematical statement is **True**. If both compound simple mathematical statements are **false,** then the truth value of the compound mathematical statement is **"false"**.

If both the component simple mathematical statements of a compound mathematical statement formed by using the connective "or" are true, then the "or" is called **Inclusive** "or". Again if one is true and other is false, then the "or" used in compound mathematical statement is called Exclusive "or".

Example 58 : Let p: Rhombus is a quadrilateral.

q: Rhombus is a parallelogram.

Here p & q both are simple mathematical statements and both are true.

r: Rhombus is a quadrilateral or a parallelogram.

Hence r is a compound mathematical statement which is obtained by connecting p and q with the connective "or". Since both p and q are true, the truth value of r is **"True"** and here "or" is **Inclusive** "**or**".

Example 59 : Let p: 85 is divisible by 7.

q: 85 is divisible by 5.

Here p & q both are simple mathematical statements. p is false but q is true.

r: 85 is divisible by 7 or it is divisible by 5.

r is a compound mathematical statement formed by connecting p and q using the connective "or". Since p is false but q is true, the truth value of r is "true" and the "or" used here is **Exclusive "or".**

Example 60 : Let p: Two straight lines intersect at a point.

Q: Two straight lines are parallel.

Here both p and q are simple mathematical statement. If p is true, then q is false or if p is false, then q is true but p and q cannot be both true or cannot be both false. Only one of p and q is true. So the truth value of r is **true** where

r: Two straight lines either intersect at a point or they are parallel.

Here "or" use is **Exclusive "or".**

Implications:

A compound mathematical statement is formed connecting two simple mathematical statements using the connecting words "if – then", "only if" and "if and only if". These connecting words are called **Implications.**

(i) The word "if – then":

Let p and q be two simple mathematical statements. if a compound mathematical statement is formed with p and q using "if p then q" – then its meaning is "if p is true then q must be true". Symbolically it is written as $p \Rightarrow q$ of $p \Rightarrow q$. (We read this as p implies q).

Example 61 : If a number is divisible by 6, then it must be divisible by 3". It is compound mathematical statement.

p: A number is divisible by 6

q: The number is divisible by 3.

Here if p is true, then q must be true. But if p is not true i.e., a number is not divisible by 6 then we cannot say that it is not divisible by 3. For example: 123 is not divisible by 6 but it must be divisible by 3.

Here p is called "sufficient condition" for q and q is called the "necessary condition" for p.

If p is true then q is false then $p \Rightarrow q$ is false.

Regarding the truth value of "if p then q" i.e., $p \Rightarrow q$ (the validity of $p \Rightarrow q$) the following is kept in mind:

(i) If p is true and q is true, then the statement $p \Rightarrow q$ is true.
(ii) If p is false and q is true, then the statement $p \Rightarrow q$ is true.
(iii) If p is false and q is false, then the statement $p \Rightarrow q$ is true.
(iv) If p is true and q is false, then the statement $p \Rightarrow q$ is false.

Example 62 : "If 42 is divisible by 7, then sum of the digits of 42 is divisible by 7". – It is a compound mathematical statement.

p: 42 is divisible by 7. q: The sum of the digits of 42 is divisible by 7.

Here p is true but q is not true.

$\therefore$ $p \Rightarrow q$ is false.

$\therefore$ The truth value of the given statement is false.

Example 63 : "If 123 is divisible by 3, then the sum of the digits of 123 is not divisible by 3". – It is a compound mathematical statement.

p: 123 is divisible by 3. q: the sum of the digits of 123 is not divisible by 3.

Here p is true and q is not true.

• $p \Rightarrow q$ is false.

• The truth value of the given statement is false.

Example 64 : "If anybody is born in India, then he is a citizen of India". It is compound mathematical statement.

p: Anybody is born in India. q: He is a citizen of India.

Here if p is true, then q is true. So $p \Rightarrow q$ is true.

Contrapositive and converse statement:

If a compound mathematical statement is formed with two simple mathematical statements p and q using the connective "if-then" then the contrapositive and converse statements of compound statement can also be formed.

The contrapositive statement of "if p, then q" is "if – q, then – p" and the converse statement is "if q then p".

Example 65 : "If a number is divisible by 6, then it is divisible by 3". – It is a compound mathematical statement.

Its **contrapositive statement** is

"if a number is not divisible by 3, then it cannot be divisible by 6".

The Converse statement is

"If a number is divisible by 3, then it is divisible by 6".

Example 66 : "If a number is an even number, then its require is even".

Its **contrapositive statement is**

"If the require of a number is not even, then the number is not even".

The converse statement is

"If the require of a number is an even number, then the number is even".

Example 67 : Examine the truth value of the following statement using contrapositive odd integer, then both X and Y are positive odd integers". (x and y are positive integer)

Solution : P : "xy is a positive odd integer, (x, y are positive integers)".

q: "Both x and y are positive odd integers.

∴ The given statement is "If p, then q."

Its contrapositive statement is ~ q ⇒ i p that is q is false implies p is false.

Let q be false.

∴ q is false ⇒ x and y both are not positive integers ⇒ at least one x and y is an even positive integer.

Let x be an even positive integer and x = 2m, m is any positive integer.

∴ xy = 2my = 2 (my) which is a positive even integer.

So xy is not a positive odd integer i.e., p is not true or p is false.

∴ q is false ⇒ p is false or ~ q is true ⇒ ~p is true

∴ The given statement is true.

(ii) The word "only if":

Let p and q be two given simple mathematical statements. If a compound mathematical statement is formed with p and q using the connective word "only if" then it implies that p only if q that is p happens only if q happens.

Example 68 : "The triangle ABC, will be equilateral only if AB = BC = CA."

Here, p: The triangle ABC is equilateral.

q: In the triangle ABC, AB = BC = CA.

Example 69 : "A number is an even integer only if the number is divisible by 2."

Here, p : A number is an even integer.

q : The number is divisible by 2.

N.B : If the implication for a compound mathematical statement contains "if-then" or "only-if" then the statement is called **conditional statement.** "if p, then q" – here **p is called antecedent** and **q is called consequent.**

Example 70 : Obtain the truth value of

(i) If 5 + 6 = 11, then 11 – 6 = 5.

(ii) If 5 + 8 = 12, then 12 + 8 = 20.

(iii) If 6 + 9 = 14, then 14 – 7 = 8

(iv) If 7 + 8 = 15, then 8 – 7 = 2

Solution:

(i) Since p: 5 + 6 = 11 is true and q: 11 – 6 = 5 is true, so p ⇒ q i.e., the given statement is true.

(ii) Since p:5 + 8 = 12 is false and q: 12 + 8 = 20 is true, so p ⇒ q i.e., the given statement is true.

(iii) Since p:6 + 9 = 14 is false and q: 14 – 7 = 8 is false, so p ⇒ q i.e., the given statement is true.

(iv) Since p: 7 + 8 = 15 is true and 8 – 7 = 2 is false, so p ⇒ q i.e., the given statement is false.

(iii) **<u>The word "if and only if"</u>:**

When a compound mathematical statement is formed with two simple mathematical statements using the connecting words "if and only if" then the statement is called **<u>Biconditional statement.</u>**

Let p and q be two simple mathematical statements. The compound statement formed with p and q using "if and only if" then the biconditional statement can be written symbolically as $p \Rightarrow q$ and $q \Rightarrow p$ or $p \Leftrightarrow q$.

In short "if and only if" is written as "iff".

The biconditional statement $p \Leftrightarrow q$ is true only when both p and q are true or both p and q are false.

Example 71 : Two triangles are congruent if and only if the three sides of one triangle are equal to the three sides of the other triangle."

This statement can be written as:

(i) If two triangles are congruent, then the three sides of one triangle are equal to the three sides of the other triangle.

(ii) If the three sides of one triangle is equal of the three sides of the other triangle, then the two triangles are congruent.

Let p: Two triangles are congruent.

q: Three sides of one triangle are equal to the three sides of the other triangle.

From (i), we get $p \Rightarrow q$ and from (ii) we get $q \Rightarrow p$.

So the given statement is the combination of both $p \Rightarrow q$ and $q \Rightarrow p$.

Here, $p \Rightarrow q$ is true and $q \Rightarrow p$ is true (But p and q both are false). So the given statement is true because p, q both are false.

Quantifiers :

In some mathematical statements some phrases like "There exists", "For all" (or for every)are used. These are called **Quantifiers**.

For example "There exists a natural number such that $x + 6 > 9$" ; "There exists a quadrilateral whose diagonals bisect each other" ; "For all natural numbers x, $x > 0$" ; "For every real number $x \neq 0$, $x^2 > 0$".

In the above statements "There exists", "For all", "For every" etc phrases are Quantifiers.

"There exists", "For some", "For at least" are called Exitential Quantifier and they are expresses as $\exists$. "For all", "For every" are called Universal Quantifiers and they are expressed by the symbol $\forall$.

Example 72 : Indicate the Quantifiers from the following statements and write the truth value in case ;

(i) For every natural number x, $x + 1 > 0$

(ii) For at least one natural number x, $x \in$ A where A= {–1,2,3,0,–3'}

(iii) There exists a natural number n, $n - 2 > 5$.

(iv) For all real number x, $x^2 > 0$.

Solution :

(i) The quantifier is "For every". The truth value of the statement is "true" because for any natural number x, $x+1\ 0$ is always true.

(ii) "For at least" is the quantifier. The truth value of the statement is "true" because $2 \in A$, $3 \in A$ and 2, 3 are natural numbers.

(iii) "There exists" is the quantifier. The truth value of the statement is "true" because for any natural number $n \geqslant 8$, the relation $n-2 \quad 5$.

(iv) "For all" is the quantifier. The truth value of the statement is "True", because x = 0 is a real number and $x^2 \rangle 0$.

Example 73 : Using Quantifiers, express the following equations into a statement :

(i) $n+2 > n, n \in N$. (ii) $x^2 < 0, x \in I^-$ (where I^- denotes the set of all negative integers) (iii) $x + 1 > 3, \ x \in R$.

Solution :

(i) There exists a natural number $n \in N$ such that $n+2>n$. The statement is true. The quantifier is "There exists".

(ii) For all negative integers $x \in I^-$, $x^2 < 0$. The statement is false because the square of any negative integer is greater than zero. The quantifier is "For all".

(iii) There exists a real number $x \in R$ such that $x+1>3$. The statement is true because for all real number x>2, the relation x+1>3 is true. The quantifier is "There exists".

Contradiction :

Contradiction is process by which we can test the validity of a given statement.

Let P : "If n>4, then $n^2 > 16$, where n is any real number". We shall show that P is true by contradiction process as follows :

Let n be a real number and n > 4 but $x^2 \ngtr 16$.

$\because n^2 \ngtr 16$, $\therefore n^2 \leq 16$. or, $n^2 - 16 \leq 0$ (n–4) (n+4) ≤ 0 ..... (1)

Since n > 4, $n \neq 4$ or $n-4 \neq 0$ and $n-4>0$.

So from (1) we get $n+4 \leq 0$ or $n \leq -4$. It is not possible because n > 4. So our assumption must be wrong i.e., $n^2 \ngtr 16$ is wrong.

$\therefore n^2 > 16$.

Self Examination Questions

1. Examine whether the following sentences are mathematical statements or not (give reasons) :

   (i) The sun is a star (ii) Go to the market. (iii) Who is the chief-minister of West Bengal? (iv) The prime factors of 15 are 3 and 5. (v) May God bless you! (vi) How nice the building is! (viii) $x^2–x+6=0$

   (viii) The roofs of $2x^2–3x–5 = 0$ are 1 and $-\frac{5}{2}$.

Ans. (i) Mathematical Statement (ii) No (iii) No (iv) Mathematical statement (v) No (vi) No (vii) No (viii) Mathematical statements

2. Write the truth value of each of the following sentences and comment whether a mathematical statement or not :

   (i) Tomorrow is Wednesday. (ii) Every rectangle is a square (iii) $\sqrt{2}$ is an irrational number (iv) Alas ! I am undone. (v) There are 31 days in the month of July and August in each year (vi) Mathematics is an interesting subject.

Ans. (i) No. truth value; not a mathematical statement (ii) False; mathematical statement (iii) True; mathematical statement (iv) No truth value; not a mathematical statement (v) "True"; mathematical statement (vi) No truth value; not a mathematical statement.

3. Write the negation of each of the following statements :

(i) Australia is a continent. (ii) The length of the sides of each parallelogram are equal. (iii) $\sqrt{3}$ is a rational number. (iv) The difference between 12 and 28 is 14. (v) All rhombuses are parallelogram.

Ans. (i) Australia is not a continent or it is not true that Australia is a continent. (ii) The length of the sides of each parallelogram are not equal or it is not true that the length of the sides of each parallelogram are equal (iii) $\sqrt{3}$ is not a rational number or it is not true that $\sqrt{3}$ is a rational number. (iv) The difference between 12 and 28 is not 14 or it is not true that the difference between 12 and 28 is 14. (v) All rhombuses are not parallelogram or it is not ture that all rhombuses are parallelogram.

4. Identify the simple and compound mathematical statements. Write the components of compound mathematical statements. Write the truth value in each case. (i) The diagonals of a parallelogram are not equal (ii) x is a real number and 2x + 5 is a rational number (iii) 42 is an even integer and it is divisible by 2, 3, 7. (iv) London is a big city it is the capital of England. (v) The moon is a star.

Ans. (i) Simle mathematical statement; true (ii) compound mathematical statement; x is a real number; 2x+5 is a rational number; false (iii) compound mathematical statement; 42 is an even integer, 42 is divisible by 2, 42 is divisible by 3, 42 is divisible by 7 ; ture (iv) compound mathematical statement; London is a big city, London is the capital of India; true. (v) mathematical statement; false.

5. Write the compound mathematical statement using the connective "and" and then write the truth value :

(i) 7 is a prime number 7 is an odd integer.

(ii) The earth is a planet. The earth moves round the sun.

(iii) Vellor belongs to India. Vellor is the capital of Tamilnadu.

(iv) All integers are positive and negative; false.

Ans. (i) 7 is a prime number and it is an odd ingeter; true

(ii) The earth is a planet and it moves round the sun; true

(iii) Vellor belongs to India and it is the capital of Tamilnadu; false

(iv) All integers are positive and negative; false.

6. Using the connecting word "or" write the compound statement in each case and write the truth value and the type or "or".

(i) Two sides of an isosceles triangle are equal. Two angles of an isosceles triangle are equal.

(ii) Two straight lines in a plane intersect at a point. Two straight lines in a plane are parallel.

(iii) 2 is a root of the equation $x^2 - 5x + 6 = 0$. 3 is a root of the equation $x^2 - 5x + 6 = 0$.

(iv) $\sqrt{5}$ is a rational number. $\sqrt{5}$ is an irrational number.

Ans. (i) Two sides or two angles of an isosceles triangles are equal; true; Inclusive "or"

(ii) Two straight lines in a plane interest or parallel; true; Exclusive "or".

(iii) The root of the equation $x^2 - 5x + 6 = 0$ is 2 or 3; true; Inclusive "or"

(iv) $\sqrt{5}$ is a rational number or an irrational number; true; Exclusive "or".

7. Using the connecting word "if then" write the compound statement and its truth value:
   (i) A number is divisible by 15. It is divisible by 3.
   (ii) A quadrilateral is a rhombus. Its all sides are equal.
   (iii) Jhon is born in India. He is an Indian.
   (iv) 256 is an even number. Its square root is an even integer.

Ans. (i) If a number is divible by 15, then the number is divisible by 3; true (ii) If a quadrilateral is a rhombus, then its all sides are equal; true (iii) If Jhon is born in India, then he is an Indian; true (iv) If 256 is an even number, then its square root is an even integer; true.

8. Using the connecting word "only if" form the compound statement.
   (i) I will not go out. It rains
   (ii) Rita will pass the examination. She reads well.
   (iii) He succeeds in every sphere of life. He works hard.
   (iv) A quadrilateral is a rectangle. The diagonals of the quadrilateral bisect each other.

Ans. (i) I will not go out only if it rains. (ii) Rita will pass the examination only if she reads well. (iii) He succeeds in every sphere of life only if he works hard. (iv) A quadrilateral is a rectangle only if its diagonal bisect each other.

9. Using the connective "if any only if" form the compound statement and write its truth value.
   (i) Two triangles are equiangular. The corresponding sides of the triangles are proportional
   (ii) A racer wins the race. He runs fast.
   (iii) A number is divisible by 3. The sum of the digits of the number is not divisible by 3.
   (iv) The birds have wings. The trees have wings.

Ans. (i) Two triangles are equiangular if and only if their corresponding sides are proportional; True (ii) A racer wins the race if and only if he runs fast; True (iii) A number is divisible by 3 if and only if the sum of digits is not divisible by 3; False (iv) The birds have wings if and only if the trees have wings; False.

10. Write the contrapositive and converse statements of the followings :
   (i) If you drink milk, then you will be strong.
   (ii) If a triangle is equilateral, then its all sides are equal.
   (iii) If you are a graduate, then you are entitled to get this job.
   (iv) If a number is an even integer, then its square is divisible by 4.

Ans. (i) If you are not strong, then you do not drink milk; If you are strong, then you drink milk.

(ii) If all sides of a triangle are not equal, then the triangle is not equilateral; If all sides of a triangle are equal, then the triangle is equilateral.

(iii) If you are not entitled to get this job then you are not a graduate, If you are entitled to get this job, then you are a graduate.

(iv) If the square of a number is not divisible by 4, then the number is not an even integer; If the square of a number is divisible by 4, then the number is an even integer.

11. Find the truth value of each of the following statements :

(i) If 7+6 = 13, then 14 – 9 =5. (ii) If 9 + 11 = 21, then 5 – 6 = –1. (iii) If 40 ÷ 5 = 9, then 5 – 13 4 (iv) If 3 + 7 = 10, then 5 + 2 = 8.

Ans. (i) True, (ii) True, (iii) True (iv) False

12. If p : "You are a science student". Q : "y on study well" be two given simple statements, then express each of the following symbolic statement into sentences :

(i) $q \Rightarrow p$ (ii) $p \Rightarrow q$ (iii) $\sim p \Rightarrow \sim q$ (iv) $\sim q \Rightarrow \sim p$.

Ans. (i) If you study well, then you a science student. (ii) If you are a science student then you study well (iii) If you are not a science student, then you do not study well (iv) If you do not study well, then you are not a science student.

13. Write the contrapositive and converse statements of $\sim q \Rightarrow p$.

Ans. Contrapositive statement : $\sim p \Rightarrow q$; converse statement : $p \Rightarrow (\sim q)$.

14. Write the contrapositive statement of the contrapositive statement of $p \Rightarrow q$.

Ans. $p \Rightarrow q$.

15. Identify the Quantifiers from the following statements :

(i) There exists a quadrilateral whose all sides are equal.

(ii) For all real number x, $x > 0$.

(iii) For at least one national number n, $n \in A$ where A = {–1, 0, 3, 5}.

Ans. (i) There exists (ii) For all (iii) For at least.

16. Using Quantifiers, express the following symbolic inequalities into statements :

(i) $N + 2 > 5,\ n \in N$ (ii) $x^2 > 0, n \in R - \{0\}$.

Ans. (i) There exists a natural number $n \in N$ such that $n + 2 > 5$.

(ii) For every real number $x \in R - \{0\}$, $x^2 > 0$.

17. If x is real number and $x^3 + 5x = 0$ then prove that x = 0 by contradiction process.

Study Note - 2

ALGEBRA

This Study Note includes

2.1 SET THEORY

In our daily life we use phrases like a bunch of keys, a set of books, a tea set, a pack of cards, a team of players, a class of students, etc. Here the words bunch, set, pack, team, class – all indicate collections of aggregates. In mathematics also we deal with collections.

A set is a well-defined collection of distinct objects. Each object is said to be an element (or member) of the set.

The symbol $\in$ is used to denote 'is an element of' or is a member of' or 'belongs to'. Thus for $x \in A$. read as x is an element of A or x belongs to A. Again for denoting 'not element of' or 'does not belongs to' we put a diagonal line through $\in$ thus $\notin$. So if y does not belong to A, we may write (using the above symbol), $y \notin A$

e.g. If V is the set of all vowels, we can say $e \in V$ and $f \notin V$

Methods of Describing a Set.

There are two methods :

1. Tabular Method (*or* Roster Method)
2. Selector Method (*or* Rule Method *or* Set Builder Method)

Tabular Method or Roster Method :

A set is denoted by capital letter, i.e. A, B, X, Y, P, Q, etc. The general way of designing a set is writing all the elements (or members) within brackets ( )or { } or [ ] . Thus a set may be written again as A = { blue, green, red}. The order of listing is not important. Further any element may be repeated any number of times without disturbing the set. The same set A can be taken as A = {blue, green, red, red, red}.

Selector Method (or Rule Method or Set Builder Method) :

In this method, if all the elements of a set possess some common property, which distinguishes the same elements from other non-elements, then that property may be used to designate the set. For example, if x (an element of a set B) has the property having odd positive integer such that 3 is less than equal to x and x is less than equal to 17, then in short, we may write,

B = {x : x is an odd positive integer and $3 \le x \le 17$}

In Tabular method, B = { 3, 5, 7, 9, 11, 13, 15, 17}

Similarly, C = {x : x is a day beginning with Monday}.

[*Note* 1. ':' used after x is to be read as 'such that'. In some cases 'I' (a vertical line) is used which is also to be read 'such that'.

2. If the elements do not possess the common property, then this method is not applicable]

2.1.1. TYPES OF SETS :

1. Finite Set

It is a set consisting of finite number of elements.

e.g. : A = {1, 2, 3, 4, 5}; B = { 2, 4, 6, ….., 50}; C = { x : x is number of student in a class}.

2. Infinite Set

A set having an infinite number of elements is called an Infinite set.

e.g. : A = { 1, 2, 3, …..} B = { 2, 4, 6, ……}

C = { x : x is a number of stars in the sky}.

3. Null or empty or Void Set

It is a set having no element in it, and is usually denoted by ϕ (read as phi) or { }. **As for Example :** The number of persons moving in air without any machine. A set of positive numbers less than zero.

A = { x : x is a perfect square of an integer $5 < x < 8$}.

B = { x : x is a negative integer whose square is – 1}

Remember : (i) $\phi \neq \{\phi\}$, as $\{\phi\}$ is a set whose element is ϕ.

(ii) $\phi \neq \{0\}$ is a set whose element is 0.

4. Equal set

Two sets A and B are said to be equal if all the elements of A belong to B and all the elements of B belongs to A i.e., if A and B have the same elements.

As for example : A = { 1, 2, 3, 4} : B = {3, 1, 2, 4},

or, A = {a, b, c} : B {a, a, a, c, c, b, b, b, b}.

[*Note :* The order of writing the elements or repetition of elements does not change the nature of set]

Again let A = { x : x is a letter in the word STRAND}

B = { x : x is a letter in the word STANDARD}

C = { x : x is a letter in the word STANDING}

Here A = B, $B \neq C$, $A \neq C$

5. Equivalent Set

Two sets are equivalent if they have the same number of elements. It is not essential that the elements of the two sets should be same.

As for example :

A = {1, 2, 3, 4} B = { b, a, l, 1}.

In A, there are 4 elements, 1, 2, 3, 4,

In B, there are 4 elements, b, a, l,1 (one-to-one correspondence), Hence, $A \equiv B$ (symbol $\equiv$ is used to denote equivalent set)

6. **Sub-set :**

A set N is a subset of a set X, if all the elements of N are contained in/members of the larger set X.

Example

If, X = {3, 5, 6, 8, 9, 10, 11, 13}

And, N = {5, 11, 13}

Then, N is a subset of X.

That is, $N \subseteq X$ (where $\subseteq$ means 'is a subset of').

<u>Number of Subsets</u>

If, M = {a, b, c}

Then, the subsets of M are:

{a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}, {}

Therefore, the number of subsets, S = 8

And the formula, $S = 2^n$

Where,

S is the number of sets

And, n is the number of elements of the set

in the formula used to calculate the number of subsets of a given set.

So from above, M = {a, b, c}

$S = 2^n$

$= 2^3$

$= 2 \times 2 \times 2$

$= 8$

Note: Every set is a subset of itself, and the empty set is a subset of all sets.

7. **Proper Sub-set :**

If each and every element of a set A are the elements of B and there exists at least one element of B that does not belongs to A, then the set A is said to be a proper sub-set of B (or B is called super-set of A). Symbolically, we may write,

$A \subset B$ (read as A is proper sub-set of B)

And $B \subset A$ means A is a super-set of B.

If B = {a, b, c}, then proper sub-sets are {a}, {b}, {c}, {a, b}, {b,c}, {a, c}, ϕ

[Note : (i) A set is not proper sub-set of itself.

(ii) Number of proper sub-sets of a set A containing n elements is $2^n - 1$

(iii) ϕ is not proper sub-set of itself].

8. Power set :

The family of all sub-set of a given set A is known as power set and is denoted by P(A)

As for example : (i) If A = {a}, then P(A) = {{a}, ϕ,}

(ii) If A = {a, b}, then P(A) = {{a}, {b}, {a, b}, ϕ.}

(iii) If A = {a, b, c}. P (A) = {{a}, {b}, {c}, {a, b}, {b, c}, {a, c}, {a, b, c}, ϕ.}

Thus when the number of elements of A is 1, then the number of sub-sets is 2; when the number of elements of A is 2; then the number sub-sets is 4 = 2^2 and when it is 3, the number of sub-sets is 8 = 2^3. So, if A has n elements, P(A) will have 2^n sub-sets.

9. Universal Set :

In mathematical discussion, generally we consider all the sets to be sub-sets of a fixed set, known as Universal set or Universe, denoted by U. A Universal set may be finite or infinite.

As for example :

(i) A pack of cards may be taken as universal set for a set of diamond or spade.

(ii) A set of integers is Universal set for the set of even or odd numbers.

10. Cardinal Number of a set :

The cardinal number of a finite set A is the number of elements of the set A. It is denoted by n{A).

e.g. : If A = {1, m, n}, B = {1, 2, 3} then n(A) = n(B)

2.1.2 Venn Diagram :

John Venn, an English logician (1834 – 1923) invented this diagram to present pictorial representation. The diagrams display operations on sets. In a Venn diagram, we shall denote Universe U (or X) by a region enclosed within a rectangle and any sub-set of U will be shown by circle or closed curve.

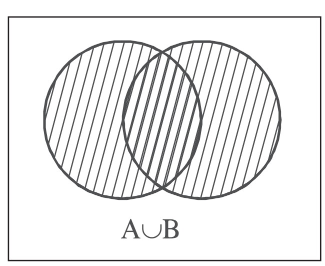
Overlapping Sets :

If two sets A and B have some elements common, these are called overlapping sets.

e.g. : If A = {2,5,7,8} and B = {5, 6, 8}, they are called overlapping sets.

Union of Sets

If A and B are two sets, then their union is the set of those elements that belong either to A or to B (or to both).



The union of A and B is denoted symbolically as $A \cup B$ (read as A union B or A cup B).

In symbols, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

As for example :

(i) Let A = {1, 2, 3, 4, 5}, B = {2, 3, 4, 6, 7}, C ={ 2, 4, 7, 8, 9}.

Then $A \cup B$ = {1, 2, 3, 4, 5, 6, 7}

and $B \cup A$ = {1, 2, 3, 4, 5, 6, 7}

$\therefore$ $A \cup B = B \cup A$ (commutative law)

Again $(A \cup B) \cup C$ = {1, 2, 3, 4, 5, 6, 7, 8, 9}

$(B \cup C)$ = {2, 3, 4, 6, 7, 8, 9}

$A \cup (B \cup C)$ = {1, 2, 3, 4, 5, 6, 7, 8, 9}

$\therefore (A \cup B) C = A \cup (B \cup C)$ (associative law)

(ii) If A = {a, b, c, d}, B = {0}, C = ϕ, then

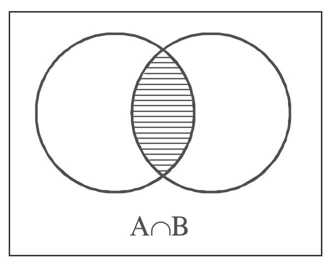
$A \cup B$ = {0, a, b, c, d},

$A \cup C$ = {a, b, c, d} = A and $B \cup C$ = {0}

Union of sets may be illustrated more clearly by using Venn Diagram as above.

The shaded region indicates the union of A and B i.e. $A \cup B$

Intersection of Sets



If and B are two given sets, then their intersection is the set of those elements that belong to both A and B, and is denoted by $A \cap B$ (read as A intersection of B or A cap B).

As for example :

(i) For the same sets A, B, C given in above example:

$A \cap B$ = { 2, 3, 4} here the elements 2,3,4, belong both to A and B; and

$B \cap A$ = { 2, 3, 4}

$\therefore$ $A \cap B = B \cap A$ (commutative law).

$(A \cap B) \cap C$ = {2, 4}

$(B \cap C)$ = {2, 4, 7}, $A \cap (B \cap C)$ = {2, 4}

$\therefore$ $(A \cap B) \cap C = A \cap (B \cap C)$ (associative law)

(ii) For the sets A, B, C given in example (ii) above,

$A \cap B = \phi$, $B \cap C = \phi$, $A \cap C = \phi$.

Intersection of two sets A and B is illustrated clearly by the Venn Diagram as given above

The shaded portion represents the intersection of A and B i.e., $A \cap B$

Disjoint Sets :

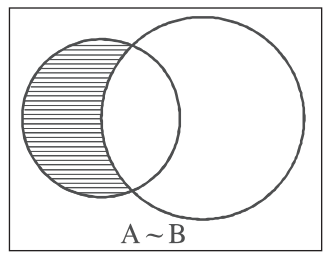
Two sets A and B are said to be disjoint if their intersection is empty, i.e., no element of A belongs to B.

e.g. : $\therefore$ A = {1,3, 5}, B = {2, 4},

$A \cap B = \phi$. Hence, A and B are disjoint sets.

Difference of two sets

If A and B are two sets, then the set containing all those elements of A which do not belong to B, is known as difference of two sets, and is denoted by the symbol A ~ B or A – B (read A difference B).



Now, A ~ B is said to be obtained by subtracting B from A.

In symbols, $A \sim B = \{x ; x \in A \text{ and } x \notin B\}$,

As for example :

(i) If A = {1, 2, 3 , 4, 5}

B = { 3, 5, 6, 7}, then A ~ B = {1, 2, 4}

(ii) If A = {x : x is an integer and $1 \le x \le 12$}, B = {x : x is an integer and $7 \le x \le 14$}

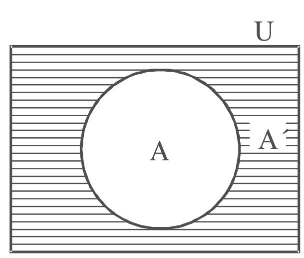
then A ~ B = {x : x is an integer and $1 \le x \le 6$},

A ~ B is represented by a Venn diagram as above :-

The shaded portion represents A ~ B.

Complement of a Set :

Let U be the universal set and A be its sub-set. Then the complement set of A in relation to U is that set whose elements belong to U and not to A.



This is denoted By A´ (= U ~ A) or A´ or A.

In symbols, $A' = \{x : A \in U \text{ and } x \notin A\}$.

We may also write : $A' = \{x : x \notin A\}$.

Remarks :

1. The union of any set A and its complement A´ is the universal set, i.e., $A \cup A' = U$.

2. The intersection of any set and its complement A´ is the null set, i.e. , $A \cap A' = \phi$.

As for example : U = {1, 2, 3, ………, 10}, A = { 2, 4, 7}

A´ (= U ~ A) = {1, 3, 5, 6, 8, 9, 10} = U, $A \cap A' = \phi$

Again (A´)´ = {2, 4, 7) = A, (i.e., complement of the complement of A is equal to A itself.

U´ = ϕ , (i.e., complement of a universal set is empty).

Again the complement of an empty set is a universal set, i.e., ϕ' = U.

If $A \subset B$ then $B' \subset A'$ for set A and B.

Complement of A is represented by shaded region.

Symmetric Difference :

For the two sets A and B, the symmetric difference is $(A \sim B) \cup (B \sim A)$

and is denoted by A Δ B (read as A symmetric difference B)

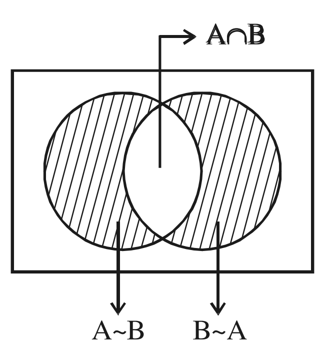
As for example : Let A = { 1, 2, 3, 4,8}, B = {2, 4, 6, 7}.

Now, A ~ B = { 1, 3, 8}, B ~ A = {6, 7}

$\therefore$ A Δ B = { 1, 3, 8} $\cup$ (6,7) = {1, 3, 6, 7, 8}

By Venn diagram :

A Δ B is represented by shaded region. It is clear that A Δ B denotes the set of all those elements that belong to A and B except those which do not belong to A and B both, i.e., is the set of elements which belongs to A or B but not to both.



Difference between :

ϕ, (0) and {ϕ}

ϕ is a null set.

{0} is a singleton set whose only element is zero.

{ϕ} is also a singleton set whose only element is a null set.

2.1.3 Properties :

1. The empty set is a sub-set of any arbitrary set A.

2. The empty set is unique.

Note :

(i) ϕ has only one subset $\{\phi\}$

(ii) $\phi \neq \{\phi\}$ but $\phi \in \{\phi\}$; $\{2\} \neq 2$.

3. The complement of the complement of a set A is the set A itself, i.e., $(A')' = A$.

SOLVED EXAMPLES

Example 1: Rewrite the following examples using set notation :

(i) First ten even natural numbers.
(ii) Set of days of a week.
(iii) Set of months in a year which have 30 days.
(iv) The numbers 3, 6, 9, 12, 15.
(v) The letters m, a, t, h, e, m, a, t, i, c, s.

Solution:

(i) A = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20} (Tabular method)
= { x : x is an even integer and $2 \leq x \leq 20$} (Select method)
(ii) A = {Sunday, Monday, ….., Saturday} (Tabular method)
= { x : x is a day in a week} (Selector method)
(iii) A = {April, June, September, November} (Tabular method)
= {x : x is a month of 30 days} (Selector)
(iv) A = { x : x is a positive number multiple of 3 and $3 \leq x \leq 15$}
(v) A = { x : x is a letter in the word mathematics}.

Example 2 : Write the following set in roster form.

(i) A = {x : x is an integer, $-3 \leq x < 7$}
(ii) B = {x : x ix an integer, $4 < x \leq 12$}

Solution:

(i) A = { –3, –2, –1, 0, 1, 2, 3, 4, 5, 6}
(ii) B = { 6, 8, 10, 12}

Example 3 : Represent the following sets in a selector method :

(i) all numbers less than 15
(ii) all even numbers

Solution:

Taking R to be the set of all real numbers in every case :

(i) {x : $x \in$ R and $x < 15$}
(ii) {x : $x \in$ R and x is a multiple of 2}

Example 4 : State **:**

(i) Is the set A = {x : x < x} a null?
(ii) Is the set B = {x : x + 4 = 4} a null?
(iii) Is the set C = { x : x is a positive number less than zero} a null?

Solution:

(i) Null, as there exists no number less than itself.
(ii) Not null, the set has an element zero.

(iii) Null, as there exists no positive number less than zero.

Example 5 : State with reasons whether each of the following statements is true or false :

(i) $\{1\} \in \{1,2,3\}$, (ii) $1 \in \{1, 2, 3\}$, (iii) $\{1\} \subset \{1, 2, 3\}$

Solution:

(i) False, {1} is a singleton and not an element of {1, 2, 3}

(ii) True, since 1 is an element and belongs to {1, 2, 3}

(iii) True, {1} is a proper sub-set of {1, 2, 3}

Example 6 : Let A = { 1, 3, {1}, {1, 3}, find which of the following statements are correct :

(i) $\{3\} \in A$, (ii) $\{3\} \subset A$, (iii) $\{\{1\}\} \in A$

Solution:

(i) Incorrect as the set {3} is not an element of A.

(ii) Correct as the set {3} is a subset of A.

(iii) Incorrect as the set {{1}} is not an element of A.

Example 7 : A = {1, 2, 3, 4, 6, 7, 12, 17, 21, 35, 52, 56}, B and C are subsets of A such that B = {odd numbers}, C = {prime numbers}.

List the elements of the set $\{x : x \in B \cap C\}$.

Solution:

$B \cap C = \{1, 3, 7, 17, 21, 35\} \cap \{2, 3, 7, 17\} = \{3, 7, 17\}$ $\therefore$ reqd. list = {3, 7, 17}

Example 8 : If S be the set of all prime numbers and M = {0, 1, 2, 3}, find $S \cap M$.

Solution:

S = { 2, 3, 5, 7, 11, ….}, M = { 0, 1, 2, 3}; $S \cap M = \{2, 3\}$

Example 9 : If A = { 1, 2, 3, 4, 5}, B = {2, 4, 5, 8}, C = {3, 4, 5, 6,7}, find $A \cup (B \cup C)$.

Solution:

$B \cup C = \{2, 3, 4, 5, 6, 7, 8\}$, $A \cup (B \cup C) = \{1, 2, \ldots\ldots\ldots, 7, 8\}$

Example 10 : If A = {1, 2, 3}, and B = {2, 3, 4}; find $(A–B) \cup (B–A)$

Solution:

A–B = {1}, B–A = {4}, $(A–B) \cup (B–A) = \{1, 4\}$

Example 11 : If S is the set of all prime numbers, M = { x : o ≤ x ≤ 9}

exhibit (i) $M – (S \cap M)$ (ii) $M \cup N$, N = {0, 1, 2, ...20}

Solution:

S = {2, 3, 5, 7, 11, 13, ...}, M = {0, 1, 2, ...8, 9}

(i) $S \cap M = \{2, 3, 5, 7\}$

(ii) $M \cap N = \{0, 1, \ldots, 20\}$

Example 12 : Find $A \cap B$, if A = {letter of word ASSASSINATION}

and B = {letter of word POSSESSION}

Solution:

$A \cap B$ = {SSSSION} as common letters.

OBJECTIVE QUESTIONS

1. Given : A = {1, 2, 3, 4, 5}, B = {2, 4, 6}, C = {3}, D = {0, 1, 2,.....,9]. Find :
(i) $A \cup C$, (ii) $A \cup (B \cup C)$, (iii) $B \cap C$, (iv) $C \cap D$ (v) $A \cap (B \cup C)$, (vi) $(A \cap B) \cap C$,
(vii) $A \Delta B$. [Ans. (i) {1, 2, 3, 4, 5}, (ii) {1, 2,....,6}, (iii) {ϕ}, (iv) {3}, (v) {1, 2,......, 6}, (vi) (ϕ), (vii) {1,3, 5,6}]

2. Given, U (universal) = {0, 1, ...,9}, A = {2, 4, 6}, B = {1, 3, 5, 7}, C = {6, 7}. Find
(i) $A' \cap B$. (ii) $(A \cup B) \sim C$, (iii) $(A \cup C)'$ (iii), $(A \cap U) \cap (B \cap C)$.
[Ans. (I) {1, 3, 5, 7}, (ii) {1, 2, 3, 4, 5,} (iii) { 0, 1, 3, 5, 8, 9}, (iv) {6}]

3. If S be the set of all prime numbers, M = {0, 1, 2, ...} Exhibit :
(i) $S \cap M$, (ii) $(S \cap M)$ [Ans. (i) {2, 3, 5, 7}, (ii) {0, 1, 4, 6, 8, 9}]

4. *Let* A = {a, b, c}, B = {d}, C = {c, d}, D = {a, b, d}, E = {a, b}.
Determine if the following statements are true?
(i) $E \subset A$, (ii) $B \subset C$, (iii) $A \subset D$, (iv) $C \subset D$, (v) $E = C$, (vi) $B \supset C$, (vii) $A \sim D$.
[Ans. (i) T, (ii) T, (iii) F, (iv) F, (v) F, (vi) F, (vii) T]

5. Determine which of the following sets are same :
A = { 5, 7, 6,} B = {6, 8, 7}, C = { 5, 6, 7}
D = {x : x is an integer greater than 2 but less than 6}
E = {1, 2, 3, 4, 5, 6} F = {3, 4, 5} [Ans. A = C; D = F]

6. Fill up the blanks by appropriate symbol $\in$, $\notin$, $\subset$, $\subseteq$, $\supset$, =
(i) 3 : .....(3, 4) $\cup$ (4, 5, 6)
(ii) (6) ..... (5, 6) $\cap$ (6, 7, 8)
(iii) {3, 4, 5}......{2, 3, 4} $\cup$ {3, 4, 5}
(iv) (a, b).....(a)
(v) 4....(3, 5) $\cup$ (5, 6, 7)
(vi) (1, 2, 2, 3) ....(3, 2, 1) [Ans. (i) $\in$ (ii) $\subseteq$ (iii) $\subset$ (iv) $\notin$ (v) $\subseteq$ (vi) =]

7. Find the power set P(A) of the set A = {a, b, c} [Ans. See text part]

8. Indicate which of the sets is a null set?
X = (x : x^2 = 4, 3x = 12), Y = (x : x + 7= 7], Z = (x : $x \neq x$). [Ans. Yes, No, Yes]

9. If U = {x : x is letter in English alphabet)
V = (x : x is a vowel)
W = (x : x is a consonant)
Y = (x : x is e or any letter before e in alphabet)
Z = (x : x is e or any one of the next four letters)
Find each of the following sets :
(i) $U \cap V$, (ii) $V \cap Y$, (iii) $Y \cap Z$, (iv) $U \cap W'$, (v) $U \cap (W \cap V)$.
[Ans. (i) x : x is vowel, (ii) (a, e), (iii) (e), (iv) same as (i), (v) ϕ]

10. Given A = (x : x $\in$ N and x is divisible by2)

B = (x : x $\in$ N and x is divisible by 2)

C = (x : $\in$ N and x is divisible by 4)

Describe A $\cap$ (B $\cap$ C) [Ans. x : x $\notin$ N and x is divisible by 12]

11. A = (x : $\in$ N and x $\leq$ 6)

B = (x : x $\in$ N and 3 $\leq$ x $\leq$ 8)

U = (x : x $\in$ N and x $\leq$ 10)

Find the elements of the following sets with remark, if any :

(i) (A $\cup$ B)´, (ii) A´ $\cap$ B´, (iii) (A $\cap$ B)´, (iv) A´ $\cup$ B´ [Ans. (I) (9, 10), (ii) (9, 10),

(iii) {1, 2, 7, 8, 9, 10) , (iv) { 1, 2, 7, 8, 9,, 10}

12. (a) Which the following sets is the null set ϕ ? Briefly say why?

(i) A = (x : x > 1 and x < 1). (ii) B = (x : x + 3 = 3), (iii) C = (ϕ) [Ans. (i)]

(b) Which of the following statements are correct /incorrect?

3 $\subseteq$ (1, 3, 5); 3 $\in$ (1, 3, 5); (3) $\subseteq$ (1, 3, 5) (3) $\in$ (1, 3, 5)

[Ans. 2 nd and 3 rd are correct]

13. Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9,10} be the universal set. Suppose A = {1, 2, 3, 4, 5,6} an and B = {5, 6, 7} are its two subsets. Write down the elements of A – B and A $\cap$ B'.

[Ans. {1, 2, 3, 4,}; {1, 2, 3, 4}]

14. Let S = {1, 2, 3, 4, 5} be the universal set and let A = {3, 4, 5} and B = {1, 4, 5} be the two of its subsets.

Verify : (A $\cup$ B)´ = A´ $\cup$ B´

15. If S = { a, b, c, d, e, f} be the universal set and A, B, C, are three subsets of S, where A = { a, c, d, f}, B $\cap$ C = { a, b, f} find (A $\cup$ B) $\cap$ (B $\cup$ C) and B´ $\cap$ C´

[Ans. { a, b, c, d, f}; {c, d, e}]

16. Let A = {a, b, c} B= (a, b), C = (a, b, d), D = (c, d), E = (d). State which of the following statements are correct and give reasons :

(i) B $\subset$ A

(ii) D $\not\subset$ E

(iii) D $\subset$ B

(iv) {a} $\subset$ A [Ans. (i) and (iv) are correct]

17. List the sets, A, B and C given that :

A $\cup$ B = { p, q, r, s}; A $\cup$ C = {q, r, s, t}; A $\cap$ B = {q, r}; A $\cap$ C = {q, s}

[Ans. A = {q, r, s}, B = {p, q, r}, C = {q, s, t}]

18. If A = {2, 3, 4, 5}, B = {1, 3, 4, 5, 6, 7} and C = {1, 2, 3, 4} verify that :

A $\cap$ (B $\cup$ C) = (A $\cap$ B) $\cup$ (A $\cap$ C)

(Hints : B $\cup$ C = {1, 2, 3, 4, 5 , 6, 7}; A $\cap$ (B $\cup$ C) = {2, 3, 4, 5} & etc.)

Number of Elements in a set :

In a finite set, if operations are made, some new subsets will be formed. In this section we will find the values of these new subjects. Since A is a finite set, we shall denote it by n (A) for the finite elements in A, which may be obtained by actual counting. But for unions of two or more sets, we have different formulae :

1. For union of Two sets :

For two sets A and B which are not disjoint.

$n (A \cup B) = n (A) + n (B) - (A \cap B)$

2. For Union of Three Sets :

Let A, B and C be the three sets (no mutually disjoint); then

$n (A \cup B \cup C)= n (A) + n (B) + n (C) - n (A \cap B) - n(C \cap A) - n(B \cap C) + n (A \cap B \cap C)$

Note : (i) $n (A \cap B') = n (A)- n(A \cap B)$ (ii) $n (A \cap B)' = (A' \cup B')$

SOLVED EXAMPLES

Example 13 : In a class of 100 students, 45 students read Physics, 52 students read Chemistry and 15 students read both the subjects. Find the number of students who study neither Physics nor Chemistry.

Solution:

We know $n (A \cup B) = n (A) + n (B) n (A \cap B)$. Let A indicates Physics, B for Chemistry. Now $n (A) = 45$, $n (B) = 52$, $n(A \cap B) = 15$

So, $n (A \cup B) = 45 + 52 - 15 = 82$.

We are to find $n (A' \cap B') = n (A \cup B)' = 100 - n (A \cup B) = 100 - 82 = 18$.

Example 14 : In a class of 30 students, 15 students have taken English, 10 Students have taken English but not French. Find the number of students who have taken (i) French and(ii) French but not English.

Solution:

Let E stands for English, F for French.

$n (E \cup F) = 30$, $n (E) = 15$, $n (E \cap F') = 10$

$n (E \cup F) = n (E) + n (F) - n (E \cap F)$ ....(i)

Now $n (E \cap F') = n (E) - n (E \cap F)$

or, $10 = 15 - n (E \cap F)$, or, $n (E \cap F) = 15 - 10 = 5$

From (i) , $30 = 15 + n (F) - 5$ or, $n(F) = 20$

$n (F \cap E') = n (F) - n(F \cap E) = 20 - 5 = 15$.

Example 15 : In a class of 50 students, 15 read Physics, 20 Chemistry and 20 read Mathematics, 3 read Physics and Chemistry, 6 read Chemistry and Mathematics and 5 read Physics and Mathematics, 7 read none of the subjects. How many students read all the three subjects?

Solution:

Let A stands for Physics, B for Chemistry, C for Mathematics

Now $n (A) = 15$, $n (B) = 20$, $n (C) = 20$

$n (A \cap B) = 3$, $n (B \cap C) = 6$, $n (C \cap A) = 5$, $n (A \cap B \cap C) = ?$

and $n (A \cup B \cup C) = 50 - 7 = 43$, as 7 students read nothing.

From $n (A \cup B \cup C)$

$= n (A) + n (B) + n (C) - n (A \cap B) - n (B \cap C) - n (C \cap A) + n (A \cap B \cap C)$

or, $43 = 15 + 20 + 20 - 3 - 6 - 5 + n(A \cup B \cup C)$

or, $n(A \cap B \cap C) = 2$.

Example 16 : In a survey of 1000 families it is found that 454 use electricity, 502 use gas, 448 use kerosene, 158 use gas and electricity, 160 use gas and kerosene and 134 use electricity and kerosene for cooking. If all of them use at least one of the three, find how many use all the three fuels.

Solution:

Let us take E for electricity, G for gas, K for kerosene.

Now $n(E) = 454$, $n(G) = 502$, $n(K) = 448$.

$n(G \cap E) = 158$. $n(G \cap K) = 160$, $n(E \cap K) = 134$, $n(E \cap G \cap K) = ?$

$n(E \cup G \cup K) = 1000$

Again $n(E \cup G \cup K) = n(E) + n(G) + n(K) - n(E \cap G)$

$- n(G \cap K) - n(K \cap E) + (E \cap G \cap K)$

or, $1000 = 454 + 502 + 448 - 158 - 160 - 134 + n(E \cap G \cap K)$

$= 952 + n(E \cap G \cap K)$

or, $n(E \cap G \cap K) = 1000 - 952 = 48$.

Example 17 : In a class of 50 students appearing for an examination of ICWA, from a centre, 20 failed in Accounts, 21 failed in Mathematics and 27 failed in Costing, 10 failed both in Accounts and Costing, 13 failed both in Mathematics and Costing and 7 failed both in Accounts and Mathematics. If 4 failed in all the three, find the number of

(i) Failures in Accounts only.

(ii) Students who passed in all the three subjects.

Solution:

A = Accounts, M = Mathematics, C = Costing [No. of students failed (say)]

Now $n(A) = 20$, $n(M) = 21$, $n(C) = 27$, $n(A \cap C) = 10$, $n(M \cap C) = 13$, $n(A \cap M) = 7$ $n(A \cap M \cap C) = 4$.

(i) $n\left(A \cap \overline{M} \cap \overline{C}\right) = n(A) - n(A \cap M) - n(A \cap C) + n(A \cap M \cap C) = 20 - 7 - 10 + 4 = 7$

(ii) Total no of students failed

$= n(A) + n(M) + n(C) - n(A \cap M) - n(M \cap C) - n(A \cap C) + n(A \cap M \cap C)$

$= 20 + 21 + 27 - 7 - 13 - 10 - 4 = 42$

$\therefore$ reqd. no. of pass $= 50 - 42 = 8$.

SELF EXAMINATION QUESTIONS

1. (i) If $n(A) = 20$, $n(B) = 12$, $n(A \cap B) = 4$, find $n(A \cup B)$ [Ans. 28]

   (ii) H $n(A) = 41$, $n(B) = 19$, $n(A \cap B) = 10$, find $n(A \cup B)$ [Ans. 50]

   (iii) If $n(A) = 12$, $n(B) = 20$, and $A \chi B$, Find $n(A \cup B)$ [Ans. 20]

   (iv) If $n(A) = 24$, $n(B) = 18$ and $B \subset A$, find $n(A \cup B)$ [Ans. 24]

   [Hints. $n(A \cap B) = n(B)$ as $B \subseteq A$

   $n(A \cup B) = n(A) + n(B) - n(A \cap B) = n(A) + n(B) - n(B) = n(A) =$ etc.]

2. In a class 60 students took mathematics and 30 took Physics. If 17 students were enrolled in both the

subjects, how many students all together were in the class, who took Mathematics or physics or both, [Ans. 73]

3. In a class of 52 students, 20 students play football and 16 students play hockey. It is found that 10 students play both the game. Use algebra of sets to find out the number of students who play neither. [Ans. 24]

4. In a class test of 45 students, 23 students passed in paper first, 15 passed in paper first but not passed in paper second. Using set theory results, find the number of students who passed in both the papers and who passed in paper second but did not pass in paper first [8; 22]

5. In a class of 30 students, 15 students have taken English, 10 students have taken English but not French. Find the number of students have taken: (i) French, and (ii) French but not English. [Ans. 20,15]

6. In a class test of 70 students, 23 and 30 students passed in mathematics and in statistics respectively and 15 passed in mathematics but not passed in statistics. Using set theory result, find the number of students who passed in both the subjects and who did not pass in both the subjects. [Ans. 8; 25]
[hints : refer solved problem]

7. In a survey of 100 students it was found that 60 read Economics, 70 read mathematics, 50 read statistics, 27 read mathematics and statistics, 25 read statistics and Economics and 35 read mathematics and Economics and 4 read none. How many students read all there subjects?
[hints : refer solved problem no. 3] [Ans. 3]

OBJECTIVE QUESTIONS

1. Write the following in roster form
   (i) A = { x : x is negative odd integer, $-7 \le x \le -3$}
   (ii) B = {x : x is positive even integer, $3 < x \le 9$} [Ans. (i) –7, –5, –3 ; (ii) 4, 6, 8]

1. Represent the following in selector method
   (i) all real numbers in open interval { 1, 11}
   (ii) all real numbers in closed interval {–2, 3}
   [Ans. (i) $x \in A$, $1 < x < 11$; (ii) $x \in A$, $-2 \le x \le 3$}

2. State with reason whether each of the following statements is true or false.
   (i) $1 \subset \{1, 2, 3\}$ (ii) $\{1, 2\} \in \{1, 2, 4\}$ (iii) $\{1, 2\} \subset \{1, 2, 3\}$
   [Ans. (i) False, element is not subject of a set,
   (ii) False. Set does't belong to another set, may be subset,
   (iii) True, {1, 2} is proper subset of {1, 2, 3}]

3. A = { 1, 2, 3, 6, 7, 12, 17, 21, 35, 52, 56}, B and C are sub sets of A such that B = { odd numbers}, C = {prime numbers} list the elements of the set { $x : x \in B \cap C$} [Ans. {3, 7, 17}

4. If S be the set of all prime numbers and M = {0, 1, 2, 3}. Find $S \cap M$ [Ans. {2, 3}

5. If A = {1, 2, 3, 4 }, B = { 2, 4, 5, 8}, C = { 3, 4, 5, 6, 7}. Find $A \cup (B \cup C)$
[Ans. {1, 2, 3, 4, 5, 6, 7, 8}]

7. If A= { 1, 2, 3}, and B = {1, 2, 3, 4}. Find $(A - B) \cup (B - A)$
[Ans. {1, 4}]

2.2 INEQUATIONS

An inequality is a sort of equation. With an equation you calculate when two formulas are equal. With an inequality you calculate when one of the formulas is less (or greater) than the other formula. This of course has everything to do with the point(s) of intersection of the two formulas.

You have to know the meaning of the different signs being used.

Examples

$x + 5 > 4x + 7$.
$3x + 7 < 20x$
Solving linear inequalities is the same as solving linear equations with one very important exception —— when you multiply or divide an inequality by a negative value, it changes the direction of the inequality.

	SYMBOL	MEANING
Before we begin our example problems, refresh your memory	$<$	less than
on what each inequality symbol means. It is helpful	$>$	greater than
to remember that the "open" part of the inequality	$\leq$	less than or equal to
symbol (the larger part) always faces the larger quantity.	$\geq$	greater than or equal to

Solving single linear inequalities follow pretty much the same process for solving linear equations. We will simplify both sides, get all the terms with the variable on one side and the numbers on the other side, and then multiply/divide both sides by the coefficient of the variable to get the solution. The one thing that you've got to remember is that if you multiply/divide by a negative number then switch the direction of the inequality.

SOLVED EXAMPLES

Example 18 :

Solve the inequality $\frac{2x-3}{5} \geq \frac{x}{2} - 1$

Solution:

$$\frac{2x-3}{5} \geq \frac{x}{2} - 1$$

$$(10)\ \frac{2x-3}{5} \geq (10)\ \frac{x}{2} - 1$$

$$4x - 6 \geq 5x - 10$$

$$4x - 6 - 5x + 6 \geq 5x - 10 - 5x + 6$$

$$-x \geq -4$$

$$\frac{-x}{-1} \leq \frac{-4}{-1}$$

$$x \leq 4$$

Example 19 :

Solve the inequality $4(x + 1) < 2x + 3$

Solution :

$4(x + 1) < 2x + 3$

$4x + 4 < 2x + 3$

$4x + 4 - 2x - 4 < 2x + 3 - 2x - 4$

$2x < -1$

$\frac{2x}{2} < \frac{-1}{2}$

$x < \frac{-1}{2}$

Example 20 :

Solve the inequality $-2(m - 3) < 5(m+1) - 12$

Solution :

$-2(m - 3) < 5(m + 1) - 12$

$-2m + 6 < 5m + 5 - 12$

$-7m < -13$

$m > \frac{13}{7}$

Example 21 :

Solve the inequality $2(1 - x) + 5 \leq 3(2x - 1)$

Solution :

$2(1 - x) + 5 \leq 3(2x - 1)$

$2 - 2x + 5 \leq 6x - 3$

$10 \leq 8x$

$\frac{10}{8} \leq x$

$\frac{5}{4} \leq x$

Now, with this inequality we ended up with the variable on the right side when it more traditionally on the left side. So, let's switch things around to get the variable onto the left side. Note however, that we're going to need also switch the direction of the inequality to make sure that we don't change the answer. So, here is the inequality notation for the inequality.

$$x \geq \frac{5}{4}$$

Now, let's solve some double inequalities. The process here is similar in some ways to solving single inequalities and yet very different in other ways. Since there are two inequalities there isn't any way to get the variables on "one side" of the inequality and the numbers on the other. It is easier to see how these work if we do an example or two so let's do that.

Example 22 :

Solve the inequality $\mathbf{-6 \leq 2(x - 5) < 7}$

Solution :

$-6 \leq 2(x - 5) < 7$

The process here is fairly similar to the process for single inequalities, but we will first need to be careful in a couple of places. Our first step in this case will be to clear any parenthesis in the middle term.

$$-6 \le 2x - 10 < 7$$

Now, we want the x all by itself in the middle term and only numbers in the two outer terms. To do this we will add/subtract/multiply/divide as needed. The only thing that we need to remember here is that if we do something to middle term we need to do the same thing to BOTH of the out terms. One of the more common mistakes at this point is to add something, for example, to the middle and only add it to one of the two sides.

Okay, we'll add 10 to all three parts and then divide all three parts by two.

$$4 \le 2x < 17$$

$$2 \le x < \frac{17}{2}$$

Example 23 :

Solve the inequality $-3 < \frac{3}{2}(2 - x) \le 5$

Solution :

$$-3 < \frac{3}{2}(2 - x) \le 5$$

In this case the first thing that we need to do is clear fractions out by multiplying all three parts by 2. We will then proceed as we did in the first part.

$$-6 < 3(2 - x) \le 10$$

$$-6 < 6 - 3x \le 10$$

$$-12 < -3x \le 4$$

Now, we're not quite done here, but we need to be very careful with the next step. In this step we need to divide all three parts by -3. However, recall that whenever we divide both sides of an inequality by a negative number we need to switch the direction of the inequality. For us, this means that both of the inequalities will need to switch direction here.

$$4 > x \ge -\frac{4}{3}$$

The inequality could be flipped around to get the smaller number on the left if we'd like to. Here is that form,

$$-\frac{4}{3} \le x < 4$$

When doing this make sure to correctly deal with the inequalities as well.

Example 24 :

Solve the inequality $-14 < -7(3x + 2) < 1$

Solution :

$-14 < -7(3x + 2) < 1$

$-14 < -21x - 14 < 1$

$0 < -21x < 15$

$0 > x > -\frac{15}{21}$

$0 > x > -\frac{5}{7}$ OR $-\frac{5}{7} < x < 0$

2.3 VARIATION

DIRECT VARIATION :

If two variable quantities A and B be so related that as A changes B also changes in the same ratio, then A is said to vary directly as (or simply vary) as B. This is symbolically denoted as $A \propto B$ (read as A varies as B)

The circumference of a circle = 2 r, so circumference of a circle varies directly as the radius, for if the radius increases (or decreases), circumference also increases or decreases.

From the above definition, it follows that :

If A varies as B, then A = KB, where K is constant ($\neq 0$)

Cor. : $A \propto B$, then $B \propto A$. If $A \propto B$, then A = kB. or, $B = \frac{A}{k}$ i.e., $B \propto A$.

Inverse Variation :

A is said to vary inversely as B, if A varies directly as the reciprocal of B. i.e. if $A \propto \frac{1}{B}$.

From $A \propto \frac{1}{B}$, we have $A = K.\frac{1}{B}$ or, AB = K, K is constant.

$A = \frac{1}{B}k$. implies that, as B increases, A decreases ; or, as B decreases, A increases.

For example, for doing a piece of work, as the number of workers increases, time of completing the work decreases and conversely. Similarly, the time of travelling a fixed distance by a train varies inversely as the speed of the train.

Joint Variation :

A is said to vary jointly as B,C,D,....... If A varies directly as the product of B, C, D, ... i. e., if $A \propto$ (B. C. D ....). From $A \propto$ (B. C. D. .....) it follows A = K (B.C.D. .....), K is constant.

For example, the area of a triangle = $\frac{1}{2} \times$ base $\times$ altitude. So it follows that area varies jointly as the base and altitude. Similarly, the area of a rectangle varies jointly as its length and breadth (note, area = length $\times$ breadth)

If again A varies directly as B and inversely as C, we have

$A \propto \frac{B}{C}$ or, $A = K\frac{B}{C}$, K is constant.

For example, altitude of a triangle varies directly as the area of triangle and inversely as the base (since $\Delta = \frac{1}{2}$ a. h. So $h = \frac{2\Delta}{a}$, where Δ indicates area)

For example, the area of a triangle (Δ) varies as the base (a) when height (h) is constant and again D varies as height when base is constant. So $\Delta \propto$ a. h when both a and h vary.

Some Elementary Results :

(i) If $A \propto B$, then $B \propto A$ (ii) If $A \propto B$ and $B \propto C$, then $A \propto C$

(iii) If $A \propto B$ and $B \propto C$, then $A - B \propto C$

(iv) If $A \propto C$ and $B \propto C$, then $A - B \propto C$

(v) If $A \propto C$ and $B \propto C$, then $\sqrt{AB} \propto C$.

(vi) If $A \propto B$, then $A^n \propto B^n$.

(vii) If $A \propto B$ and $C \propto D$, then $AC \propto BD$ and $\frac{A}{C} = \frac{B}{D}$

(viii) If $A \propto BC$, then $B \propto \frac{A}{C}$ and $C \propto \frac{A}{B}$

SOLVED EXAMPLES

Example 25: Given $c \propto (ax + b)$, value of c is 3 when a = 1, b = 2 value of c is 5, when a = 2, b = 3. Find x.

Solution:

As $c \propto (ax + b)$, so $c = k(ax + b)$, k is constant.

For a = 1, b = 2, we get $c = k(x + 2)$ ….. (i)

a = 2, b = 3, we get $c = k(2x + 3)$ …… (ii)

Subtracting (i) from (ii), $0 = k(x + 1)$ or, $x + 1 = 0$ as $k \neq 0$ $\therefore$ $x = -1$.

Example 26: If the cost price of 12 kg. of rice is ₹ 10, what will be the cost of 15 kg. of rice?

Solution:

Let A (= cost) = ₹ 10, B (= quantity) = 12 kg. Now $A \propto B$ i.e., A = KB or, 10 = K. 12 or, $K = \frac{10}{12}$. Now, we are to find A, when B = 15 kg.

Again from A = KB, we have $A = \frac{10}{12}.15$ = ₹ 12.50.

Example 27 : A man can finish a piece of work, working 8 hours a day in 5 days. If he works now 10 hours daily, in how many days can he finish the same work?

Solution:

Let A (= days) = 5, B (= hours) = 8, it is clear that $A \propto \frac{1}{B}$

i.e., $A = K.\frac{1}{B}$ or, $5 = K.\frac{1}{8}$ or, K = 40.

To find A, when B = 10, we have A = $40.\frac{1}{10}$ = 4 days.

(*Partly fixed and partly variable*)

Example 28 : The publisher of a book pays author a lump sum plus an amount for every copy sold. If 500 copies are sold, the author would receive ₹ 750 and for 1350 copies ₹ 1175. How much would the author receive if 10000 copies are sold?

Solution:

Let x = lump (i.e. fixed) sum received, y = variable amount received on sale.

n = number of copies sold, so that $y \propto n$ or, y = kn, k = constant.

Again, total amount (T) = x + y = x + kn ….. (i)

So we get, 750 = x + k. 500 ……. (ii)

1175 = x +k. 1350 ………… (iii)

Solving (ii), (iii), $k = \frac{1}{2}$, x = 500. From (i) we get $T = 500 + \frac{1}{2}.n$

For n = 1,000, $T = 500 + \frac{1}{2} \times 1000$ = ₹ 5,500.

Example 29: The expenses of a boarding house are partly fixed and partly varies with the number of boarders. The charge is ₹ 70 per head when there are 25 boarders and ₹ 60 per head when there are 50 boarders. Find the charge per head when there are 100 boarders.

Solution:

Let x = fixed monthly expense, y = variable expense, n = no. of boarders.

Now $y \propto n$ or, y = k n, k is constant. The monthly expenses for 25 and 50 boarders are respectively ₹ 1,750 and ₹ 3,000.

Now total expense = fixed expenses + vairable expenses.

i.e., T = x + y = x + kn, where T = total expenses.

So, from T = x + kn, we get,

Hence, 1,750 = x + 25k ….. (1)

3,000 = x + 50k……. (2)

Subtracting (1) from (2), 25k = 1,250, or k = 50.

Again, putting the value of k in (1), we find x = ₹ 500.

Now, charge for 100 boarders = x + 100 k = 500 + 100 x 50 = ₹ 5,500

∴ Charge per head = 5,500/100 = ₹ 55.

Example 30 : As the number of units manufactured in a factory is increased from 200 to 300, the total cost of production increases from ₹ 16,000 to ₹ 20,000. If the total cost of production is partly fixed and other part varies as number of units produced, find the total cost of for production 500 units.

Solution:

Total cost (T) = fixed cost + variable cost, fixed cost = a (say) variable cost $\propto$ no. of units (n) i.e. variable cost = kn, k = constant

or, T = a + kn …….. (i)

$$\begin{array}{l} 20{,}000 = a + 300k......(i) \\ \underline{16{,}000 = a + 200k} \\ \quad 4{,}000 = \quad 100k, k = 40, \end{array}$$

From (ii) 20000 = a + 12000, a = 8000

Again for 500 units ; Total cost = 8000 + 500 × 40 = ₹ 28,000.

Example 31 : An engine without any wagons can run 24 km/hr. and its speed is diminished by a quantity varying as the square root of the number of wagons attached to it. With 4 wagons its speed becomes 20 km/hr. Find the maximum number of wagons with which the engine can move.

Solution :

Let n be number of wagons. So speed = $24 - k\sqrt{n}$, k = constant …… (i)

Again 20 = $24 - k.\sqrt{n}$ or, 2k = 4 or, k = 2

From (i), speed (s) = $24 - 2\sqrt{n}$. As n increases speed diministes,

For speed = 0, we have $0 = 24 - 2\sqrt{h}$ or, $\sqrt{n} = 12$ or, n = 144, i.e. when 144 wagons are attached engine cannot move.

∴ Engine can move with 144 – 1 = 143 wagons.

SELF EXAMINATION QUESTIONS

1. Apply the principle of variation, how long 25 men take to plough 30 hectres, if 5 men take 9 days to plough 10 hectres of land ? [Ans. $5\frac{2}{5}$ days]

2. The distance through which a heavy body falls from rest varies at the square of the time it falls. A body falls through 153 ft. in 3 secs. How far does it fall in 8 secs. And 8th sec? [Ans. 1,088ft. ; 255 ft.]

3. The time of the oscillation of a pendulum varies as the square root of its length. If a pendulum of length 40 inch oscillates once in a second, what is the length of the pendulum oscillating once in 2.5 sec.? [Ans. 250 inch.]

4. The area of a circle varies directly with the square of its radius. Area is 38.5 sq. cm. when radius of the circle is 3.5 cm. Find the area of the circle whose radius is 5.25cm. [Ans. $86\frac{5}{8}$ sq. cm.]

5. The volume of a gas varies directly as the absolute temperature and inversely as pressure. When the pressure is 15 units and the temperature is 260 units, the volume is 200 units. What will be volume when the pressure is 18 units and the temperature is 195 units? [Ans. 125units]

6. The expenses of a hotel are partly fixed and the rest varies as the number of boarders. When the number of boarders are 450, the expense is ₹ 1,800, when the number of boarders is 920, the expense is ₹ 3,210. Find the expenses per head when there are 100 boarders. [Ans. ₹ 34.50]

7. The total expenses of a hostel are partly constant and partly vary as the number of boarders. If the expenses for 120 boarders be ₹ 20000 and for 100 boarders be ₹ 17000, find for how many boarders will be ₹ 18800? [Ans. 112]

8. The expenses of a boarding house are partly fixed and partly vary with the number of boarders. The charge is ₹ 100 per head, when there are 25 boarders and ₹ 80 when are 50 boarders. Find the charge per head when there are 100 boarders. [Ans. ₹ 70]

9. The total expenses per pupil in a school consist of three parts, the first of which is constant, the second varies as the number of pupils and the third varies inversely as the number of pupils. When there are 20 pupils, the total expenses per pupil are ₹ 744 ; when there are 30 pupils the total expenses per pupil are ₹ 564 ; when there are 40 pupils, the total expenses per pupil are ₹ 484 ; find the total expenses per pupil when there are 50 pupils. [Ans. ₹ 444]

10. As the number of units manufactured increases from 6000 to 8000, the total cost of production increases from ₹ 33,000 to ₹ 40,000.

Find the relationship between y, the cost and x, the number of units made, if the relationship is linear. Hence obtain the total cost of production, if the number of units manufactured is 10000.

[Ans. y = 12000 + 7/2 x : ₹ 47000]

11. Publisher of a book pays a lump sum plus an amount for every copy he sold, to the author. If 1000 copies were sold the author would receive ₹ 2500 and if 2700 copies were sold the author would receive ₹ 5900. How much the author would receive if 5000 copies were sold?
[hints refor solved ex. 54] [Ans. ₹ 10,500]

12. The expenses of a boarding house are partly fixed and partly varies with the number of boarders. The charge is ₹ 70 per head when there are 20 boarders and ₹ 60 per head when there are 40 boarders. Find the charge per head when there are 50 boarders.
[hints refer solved ex. 55] [Ans. ₹ 58]

OBJECTIVE QUESTIONS

1. If A varies inversely with B and if B = 3 then A = 8, then find B if A = 2 [Ans. 12]

2. A is proportional to the square of B. If A = 3 then B = 16 ; find B if A = 5. [Ans. $400/9$]

3. A varies inversely with B and if B = 3 then A = 7. Find A if $B = 2\frac{1}{3}$. [Ans. 9]

4. If $x \propto y$ and x = 3, when y = 24, then find the value of y when x = 8. [Ans. 64]

5. A varies inversely with B and B = 10 when A = 2, find A when B = 4. [Ans.5]

6. If $y \propto \frac{1}{x^2}$ and x = 2 when y = 9, find y when x = 3. [Ans. 4]

7. If $A \propto B$, A = 7 when B = 21. Find the relative equation between A and B. [Ans. $A = \frac{1}{3}B$]

8. If x varies inversely with y, x = 8 when y = 3, find y when x = 2 [Ans. 12]

9. If $p \propto q^2$ and the value of p is 4 when q = 2, then find the value of q + 1 when the value of p is 9.
[Ans. – 2]

10. If $a + b \propto a - b$, prove that $a \propto b$

[hints : a + b = k (a – b), (1 – k) a = (– k – 1) b & etc.

11. If x varies as y then show that $x^2 + y^2$ varies as $x^2 - y^2$

12. If (a + b) varies as (a – b), prove that $a^2 + b^2$ varies as b^2

13. If a + 2b varies as a – 2b, prove that a varies as b

14. x and y are two variables such that $x \propto y$. Obtain a relation between x and y if x = 20. Y = 4.
[Ans. x = 5y]

2. 4 LOGARITHM

Definition of Logarithm :

Let us consider the equation $a^x = N$ $(a > 0)$ where quantity a is called the base and x is the index of the power.

Now x is said to be logarithm of N to the base *a* and is written as $x = \log_a N$

This is read as x is logarithm of N to base a.

for example : $2^4 = 16$ then $4 = \log_2 16$, $4^2 = 16$, then $2 = \log_4 16$,

$3^4 = 81$ then $4 = \log_3 81$

$9^2 = 81$ then $2 = \log_9 81$,

$2^{-3} = \frac{1}{8}$ then $-3 = \log_2 \frac{1}{8}$

Now it is clear from above examples that the logarithm of the same number with respect to different bases are different.

Special Cases :

(i) Logarithm of unity to any non-zero base is zero.

e.g. : Since $a^0 = 1$, $\log_a 1 = 0$.

Thus $\log_5 1 = 0$, $\log_{10} 1 = 0$.

(ii) Logarithm of any number to itself as base is unity.

e.g. : Since $a^1 = a$, $\log_a a = 1$.

Thus $\log_5 5 = 1$, $\log_{10} 10 = 1$, $\log_{100} 100 = 1$.

LAWS OF LOGARITHM :

LAW 1.

$\text{Log}_a (m \times n) = \log_a m + \log_a n$.

Let, $\log_a m = x$, then $a^x = m$ and $\log_a n = y$, then $a^y = n$

Now, $a^x \times a^y = a^{x+y}$, i.e., $a^{x+y} = m \times n$

or, $x + y = \log_a (m + n)$

$\therefore$ $\log_a (m \times n) = \log_a m + \log_a . n$.

Thus the logarithm of product of two quantities is equal to the sum of their logarithms taken separately.

Cor. $\text{Log}_a (m \times n \times p) = \log_a m + \log_a n + \log_a p$.

Similarly for any number of products,

LAW 2 : $\log_a \frac{m}{n} = \log_a m - \log_a n$

Thus the logarithm of quotient of any number is equal to the difference of their logarithms.

LAW 3 : $\log_a (m)^n = n. \log_a m$

Thus, the logarithm of power of a number is the product of the power and the logarithm of the number.

CHANGE OF BASE :

The relation between the logarithm of a number of different bases is given by

$\text{Log}_a m = \log_b m \times \log_a b$.

Let $x = \log_a m$, $y = \log_b m$, $z = \log_a b$, then from definition $a^x = m$, $b^y = m$, $a^z = b$.

Hence $a^x = m = b^y$

$\text{Log}_a m = \log_b m \times \log_a b$.

Cor.I. $\log_a b \times \log_b a = 1$. This result can be obtained by putting m = a in the previous result, $\log_a a = 1$.

Cor. 2. $\log_a m = \log_b m / \log_b a$.

Let $x = \log_a m$, $a^x = m$; take log to the base b we find $x \log_b a = \log_b m$.

$\therefore$ $x = \log_b m / \log_b a$.

Hence the result.

SOLVED EXAMPLES :

Example 32 : Find the logarithm of 2025 to the base $3\sqrt{5}$. .

Solution :

Let x be the required number ; then $(3\sqrt{5})^x = 2025 = 3^4.5^2 = (3\sqrt{5})^4$ $\therefore$ x = 4.

$\therefore$ 4 is the required number.

Example 33 : The logarithm of a number to the base $\sqrt{2}$ is k. What is its logarithm to the base $2\sqrt{2}$?

Solution :

Let $(\sqrt{2})^x = N$.

Since $2\sqrt{2} = 2.2^{1/2} = 2^{3/2}$

So $\sqrt{2} = (2^{3/2})^{1/3} = (2\sqrt{2})^{1/3}$

$\therefore (2\sqrt{2})^{k/3} = N$.

$\therefore$ the reqd. number is $\frac{k}{3}$.

Example 34 : Find the value of $\log_2 [\log_2 \{\log_3 (\log_3 27^3)\}]$.

Solution :

Given expression

$= \log_2 [\log_2 \{\log_3 (\log_3 3^9)\}] = \log_2 [\log_2 \{\log_3 (9 \log_3 3)\}]$

$= \log_2[\log_2\{\log_3 9\}]$ (as $\log_3 3 = 1$)

$= \log_2 [\log_2 \{\log_3 3^2\}] = \log_2 [\log_2 \{2\log_3 3\}] = \log_2 [\log_2 2] = \log_2 1 = 0$

Example 35 : If $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$, find x

Solution :

$\log_2 16 \times \log_{16} x + \log_4 16 \times \log_{16} x + \log_{16} x = \frac{21}{4}$

or $4\log_{16}x + 2\log_{16}x + \log_{16}x = \frac{21}{4}$ or $7\log_{16}x = \frac{21}{4}$

or $\log_{16} x = \frac{3}{4}$ or $x = 16^{3/4} = 2^{4\times3/4} = 2^3 = 8$.

Example 36 : If $p = \log_{10} 20$ and $q = \log_{10} 25$, find x and such that $2 \log_{10} (x + 1) = 2p - q$

Solution :

$2p - q = 2 \log_{10} 20 - \log_{10} 25 = \log_{10} (20)^2 - \log_{10} 25$

$= \log_{10} 400 - \log_{10} 25 = \log_{10} \frac{400}{25} = \log_{10} 16$

Now, $2\log_{10}(x+1) = \log_{10} 16$ or, $\log_{10} (x + 1)^2 = \log_{10} 16$ or, $(x + 1)^2 = 16 = (\pm 4)^2$

or, $x + 1 = \pm 4$

$\therefore$ $x = 3, -5$.

Example 37 : If $x = \log_{2a} a$, $y = \log_{3a} 2a$, $z = \log_{4a} 3a$, Show that : $xyz + 1 = 2yz$.

Solution :

L.H.S. $= \log_{2a} a . \log_{3a} 2a . \log_{4a} 3a + 1$

$= (\log_{10} a \times \log_{2a} 10).(\log_{10} 2a \times \log_{3a} 10).(\log_{10} 3a \times \log_{4a} 10) + 1$

$= \frac{\log_{10} a}{\log_{10} 2a} \times \frac{\log_{10} 2a}{\log_{10} 3a} \times \frac{\log_{10} 3a}{\log_{10} 4a} + 1$

$\frac{\log_{10} a}{\log_{10} 4a} + 1 = \log_{4a} a + \log_{4a} 4a = \log_{4a} (a \cdot 4a) = \log_{4a} 4a^2$

R.H.S. $= 2\log_{3a} 2a . \log_{4a} 3a = \log_{4a} (2a)^2 = \log_{4a} . 4a^2$

Hence the result.

Example 38 : Show that $\log_3 \sqrt{3\sqrt{3\sqrt{3.....\infty}}} = 1$.

Solution :

Let, $x = \sqrt{3\sqrt{3\sqrt{3}}}...$ or $x^2 = 3\sqrt{3\sqrt{3...\infty}}$

(squaring both sides)

or, $x^2 = 3x$ or, $x^2 - 3x = 0$ or, $x (x - 3) = 0$ or, $x - 3 = 0$ (as $x \neq 0$),

$\therefore$ $x = 3$

$\therefore$ given expression $= \log_3 x = \log_3 3 = 1$.

SELF EXAMINATION QUESTIONS

1. If $\frac{1}{2} \log_3 M + 3\log_a N = 1$, express M in terms of N. [Ans. $9N^{-6}$]

2. If $a^2 + b^2 = 7ab$, show that :

(i) $2 \log (a - b) = \log 5 + \log a + \log b$.

(ii) $2 \log (a + b) = \log 9 + \log a + \log b$.

(iii) $\log \frac{1}{3}(a+b) = \frac{1}{2}\{\log a + \text{lob } b\}$

3. (i) If $x^2 + y^2 = 6xy$, prove that

$2 \log (x + y) = \log x + \log y + 3 \log 2$

(ii) If $a^2 + b^2 = 23\, ab$, show that $\log\frac{1}{5}(a+b) = \frac{1}{2}\{\log a + \text{lob } b\}$

4. If $a = b^2 = c^3 = d^4$, prove that $\log_a (abcd) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$.

5. Prove : (i) $\log_2 \log_2 \log_2 16 = 1$, (ii) $\log_2 \log\sqrt{2}\ \log_3 81 = 2$

6. Prove that :

(i) $\log_b a \times \log_c b \times \log_a c = 1$

(ii) $\log_b a \times \log_c b \times \log_a c = \log_a a$.

(iii) $(1+\log_n m \times \log_{mn} x = \log_n x$

7. (a) Show that :– (i) $16\log\frac{16}{15}+12\log\frac{25}{24}+7\log\frac{81}{80}=\log 5$

(ii) $\log\frac{75}{16}-2\log\frac{5}{9}+5\log\frac{32}{243}=\log 2$

(b) Prove that

$x^{\log^y - \log^z}\ y^{\log^z - \log^x}\ z^{\log^x - \log^y} = 1$

8. Prove that (i) $\frac{\log\sqrt{27}+\log 8+\log\sqrt{1000}}{\log 120}=\frac{3}{2}$

(ii) $\frac{\log\sqrt{27}+\log 8+\log\sqrt{1000}}{\log 14400}=\frac{3}{4}$

9. Find $\log_7 \sqrt{7\sqrt{7\sqrt{7....\infty}}}$ [Ans. 1]

10. If $\frac{\log x}{y-z}=\frac{\log y}{z-x}=\frac{\log z}{x-y}$, show that xzy = 1

11. If $\log_a bc = x$, $\log_b ca = y$, $\log_c$, $ab = z$, prove that $\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1$

12. If $\frac{\log x}{l+m-2n}=\frac{\log y}{m+n-2l}=\frac{\log z}{n+l-2m}$, show that xyz = 1.

13. If $p = a^x$, $q = a^y$ and $a^4 = (p^{4y}. q^{4x})^x$ prove that $xyz = \frac{1}{2}$.

14. Prove that $\frac{\log 3\sqrt{3} + \log\sqrt{8} - \log\sqrt{125}}{\log 6 - \log 5} = \frac{3}{2}$

COMMON LOGARITHM :

Logarithm to the base 10 is called common logarithm. For numerical calculations, common logarithm is usually used. This system was first introduced by Henry Briggs.

In future, the base of the common logarithm will not be written. Thus log 4 will really $\log_{10}4$.

$10^0 = 1$ $\quad\therefore \log 1 = 0$

$10^1 = 10$ $\quad\therefore \log 10 = 1$

$10^2 = 100$ $\quad\therefore \log 100 = 2$ and so on.

Again since,

$10^{-1} = \frac{1}{10} = 0.1$ $\quad\therefore \log 0.1 = -1$

$10^{-2} = \frac{1}{10^2} = 0.01$ $\quad\therefore \log 0.01 = -2$ and so on

2.4.1 ANTILOGARITHM

If $10^x = N$, i.e., if $\log N = x$, then N is called the antilogarithm or antilog of x.

e.g : Since log 100 = 2 $\quad\therefore$ antilog 2 = 100

log 1000 = 3 $\quad\therefore$ antilog 3 = 1000

log 0.1 = 1 $\quad\therefore$ antilog 1 = 0.1

log 0.01 = 2 $\quad\therefore$ antilog 2 = 0.01 and so on.

Characteristics and Mantissa :

We know, log 100 = 2 and log 1000 = 3.

Now 517 lies between 100 and 1000.

i.e. $100 < 517 < 1000$

or $\log 100 < \log 517 < \log 1000$

or $2 < \log 517 < 3$, hence log 517 lies between 2 and 3.

In other words, log 517 = 2 + a positive proper fraction.

Again $0.001 < 0.005 < 0.01$

or $-3 < \log 0.005 < -2$, for $\log 0.001 = -3$, $\log 0.01 = -2$.

Hence log 0.005 is greater than – 3 and less than – 2 and is negative.

In other words, log 0.005 = – 3 + a positive proper fraction. Thus we see that logarithm of any number consists of two parts, an integral part (positive or negative) and a fractional part.

The integral part is called characteristics and the fractional part is mantissa.

Finding the Characteristic

(i) Let the number be greater than 1.

Any number whose integral part is of one digit only lies between 1 and 10. For example 5.7 lies between 1 and 10.

Log 5.7 lies between 0 and 1.

Now log 1 = 0 and log 10 = 1.

i.e. log 5.7 = 0 + a proper fraction.

Hence the characteristic of such numbers is 0. Again any number consisting of two digits in the integral part lies between 10 and 100.

i.e., 11, 60.7, 75.1, 92.9 etc.

Now log 11 lies between 1 and 2 ($\because$ log 10 = 1, log 100 = 2)

i.e. log 11 = 1 + a positive proper fraction.

Hence the characteristic of such numbers is 1.

Similarly, any number consisting of three digits in the integral part lies between 100 and 1000. Therefore, its logarithm lies between 2 and 3.

Hence the characteristic of such numbers is 2. Thus we arrive at following rule :

Rule 1 : The characteristic of the logarithm of a number greater than 1 is positive and is less by one than the number of the digits in the integral part.

Thus the characteristics of log 234, log 2.34 are respectively 2, 0.

(ii) Let the number of less than 1 (but greater than 0) i.e., a decimal fraction.

$10^0 = 1 \quad \therefore \log 1 = 0$

$10^{-1} = 0.1 \quad \therefore \log (0.1) = -1$

$10^{-2} = 0.01 \quad \therefore \log (0.01) = -2$

$10^{-3} = 0.001 \quad \therefore \log (0.001) = -3$ and so on.

Any fraction lying between 0. 1 and 1 has no zero between the decimal point and the first significant digit. For example 0.31, 0.471.

Now 0.31 lies between 0.1 and 1.

i.e., log 0.31 lies between log 0.1 and log 1.

$\therefore$ log 0.31 lies between – 1 and 0.

i.e. log 0.31 = – 1 + a positive proper fraction.

Hence the characteristic of such numbers is – 1.

Again any fraction lying between 0.01, 0.1 has one zero between the decimal point and the first significant digit. For example, 0.031, 0.047, 0.0707 etc.

Now 0.031 lies between 0.01 and 0.1

or log 0.031 lies between – 2 and – 1. ($\therefore$ log 0.01 = – 2 log 0.1 = – 1)

$\therefore$ log 0.031 = – 2 + a positive proper fraction.

Hence the characteristic of such numbers is – 2.

Similarly, any fraction lying between 0.001 and 0.01, has two zeros between the decimal point and first significant digit. For example 0.0031, 0.0047 etc.

Now log 0.0031 lies between – 3 and – 2.

$\therefore$ log 0.0031 = – 3 + a positive proper fraction.

Hence the characteristic of such number is – 3.

Thus we arrive at the following rule :

Rule 2 : The characteristic of the logarithm of a decimal fraction is negative and is greater by one than the number of zeroes between the decimal point and the first significant digit.

Thus the characteristic of log 0.234, log 0.0234, log 0.00234 are respectively – 1, – 2, – 3

Finding the Mantissa :

Let N be any number, the where p and q are positive integers evidently a number having the same significant digit N.

Now, log $\left(N\times10^p\right)$ = log N + log 10^p = log N + p log 10 = log N + p.

log $\left(N\div10^q\right)$ = log N – log 10^q = log N – q log 10 = log N – q.

Thus we see that p is added to and q is subtracted from the characteristic of N, while the Mantissa remains unaffected, in both cases.

Hence we get the following rule :

Rule 3 : The mantissa is the same for logarithm of all numbers which have the same significant digits (i.e., the mantissa does not depend on the position of the decimal point).

$N\times10^p$ or $N\div10^q$ **for example :** Let us consider the logarithms of the numbers 234500, 23.45, 0.02345, having given log 2345 = 3.3701.

log 234500 = log (2345 × 100) = log 2345 + log 100 = 3.3701 + 2 = 5.3701.

log 23.45 = $\log\frac{2345}{100} = \log 2345 - \log 100$ = 3.3701 – 2 = 1.3701

log 0.02345 = $\log\frac{2345}{100000} = \log 2345 - \log 100000$

= $\bar{2}.3701$, where $\bar{2}$ (read as two bar)

denotes that it is equivalent to – 2, while 0.3701 is + ve.

Thus, we see that he mantissa in every case is same.

Note : The characteristic of the logarithm of any number may be + ve or –ve, but is mantissa is always + ve.

USE OF LOGARITHMIC TABLE :

It must be observed that only approximate values can be obtained from the table, correct upto 4 decimal places. The main body of the table gives the mantissa of the logarithm of numbers of 3 digits or less, while the mean difference table provides the increment for the fourth digit.

Let us find the logarithm of 23 ; evidently the characteristic is 1.

In the narrow vertical column on the extreme left of the table, we see integers starting from 23, if we move across horizontally, the figure just below 0 of the central column is 3617.

Hence, log 23 = 1.3617.

Let us now find the logarithm of 234, its characteristic evidently is 2, if we move across horizontally starting from 23 and stop just below 4 of the central column, we find the figure 3692.

Hence log 234 = 2.3692.

Lastly, to find the logarithm of 2345, we see the characteristic is 3. Now starting form 23, if we stop below 4 of the central column we get the figure 3692. Again if we move further across the same horizontal line and stop just below 5 in extreme right column of mean difference, we get figure 9. Now adding these two figures, we find 3701, i.e., (3692 + 9).

Hence log 2345 = 3.3701

Similarly we have,

log 1963 = 3.2929

log 43.17 = 1.6352

log 7.149 = 0.8542.

For number of 5 digits

Suppose we are to find the value of log 23.456. Form 4 figure logarithmic table we get.

log 23.4	=	1.3692
Difference for (4th digit) 5	=	9
Difference for (5^{th} digit) 6	=	1 1
	=	1.3702

Rule for carry over number from the difference table :

For 0 to 4, carry over 0

5 to 14, carry over 1

15 to 24, carry over 2 and so on.

USE OF ANTILOGARITHMIC TABLE :

The antilog, gives us numbers corresponding to given numbers. At first we are to find the number corresponding to given mantissa and then to fix up the position of the decimal point according to the characteristic.

For example, to find antilog 1.5426. Now from the antilog table we can see, as before, the number corresponding to the mantissa.

0.5426 is 3483 + 5 = 3488. Since the given characteristic is 1, the required number is 34.88.

Hence log 34.88 = 1.5426, since antilog 1.5426 = 34.88.

Example 39 : If log 3 = 0.4771, find the number of digits in 3^{43}.

solution:

Let $x = 3^{43}$ then $\log x = \log 3^{43} = 43 \log 3$.

or $\log x = 43 \times 0.4771 = 20.5153$. Here the characteristic in log x is 20. So the number of digits in x will be 20 + 1 = 21.

Example 40 : Given $\log_{10} 2 = 0.30103$, find $\log_{10}$ (1000/256)

solution:

$$\log_{10}\frac{1000}{256} = \log_{10}1000 - \log_{10}256 = \log_{10}10^3 - \log_{10}2^9$$

$$= 3\log_{10}10 - 9\log_{10}2 = 3 - 9\,(0.30103) = 3 - 2.70927 = 0.29073.$$

Example 41 : Find the value of : (i) 0.8176×13.64, (ii) $(789.45)^{1/8}$

solution:

(i) Let x= 0.8176 × 13.64; taking log on both sides.

Log x = log (0.8176 ×13.64) = log 0.8176 + log 13.64

$= \bar{1}.9125 + 1.1348 = -1 + 0.9125 + 1 + 0.1348$

$= 0.9125 + 0.1348 = 1.0473$

$\therefore$ x = antilog 1.0473 = 11.15.

(ii) Let $x = (789.45)^{1/8}$ or, $\log x = \frac{1}{8}\log(789.45) = \frac{1}{8}(2.8973) = 0.3622$

$\therefore$ x = antilog 0.3622 = 2.302.

Note. Procedure of finding the mantissa of 5 significant figures will be again clear from the following example:

Mantissa of 7894 (see above) is 0.8973 and for the fifth digit (i.e. for digit 5), the corresponding number in the mean difference table is the digit 2, which is less than 5 ; so 0 is to be added to the mantissa 0.8973.

It again, the corresponding mean difference number is

5 to 14, carry 1

15 to 24, carry 2 and so on.

Example 42 : Find, from tables, the antilogarithm of – 2.7080

solution:

$-2.7080 = 3 - 2.7080 - 3 = .2920 - 3 = \bar{3}.2920$

$\therefore$ antilog (– 2.7080) = antilog $\bar{3}.2920$ = 0.001959

SELF EXAMINATION QUESTIONS

1. Find the number of zeros between the decimal point and the first significant figures in :

(i) $(0.0011)^{20}$ (ii) $\left(\frac{1}{11}\right)^{100}$ (iii) $(12.4)^{-15}$ [Ans. 59 ; 104 ; 16]

2. Given log 8 = 0.931, log 9 = 0.9542 ; find the value of log 60 correct to 4 decimal 4 places.

[Ans. 0.7781]

3. Given log 2 = 0.30103, log 3 = 0.47712 ; find the value of :

(i) log 4500 (ii) log 0.015 (iii) log 0.1875. [Ans. 3.65321 ; $\bar{2}.17609$; (iii) $\bar{1}.27300$

4. Using tables find the value of :

(i) $19.66 \div 9.701$ (ii) 0.678 ´ 9.310.0234 (iii)

(iv) $\frac{1}{(1.045)^{20}}$ (v) $\sqrt[7]{\frac{1}{1.235}}$. [Ans. (i) 2.027 ; (ii) 261 ; (iii) 0.4093 ; (iv) 0.415 ; (v) 0.9703]

OBJECTIVE QUESTIONS

1. Find the value of log 64 with base 4 [Ans. 3]

2. Find the value of log 125 with base $5\sqrt{5}$ [Ans. 2]

3. Find the value of log 144 with base $2\sqrt{3}$ [Ans. 4]

4. Show that log (1 + 2 + 3) = log1 + log2 + log3

5. Show that $\log_2 \log_2 \log_2 16 = 1$

6. Show that $\log_4 \log_{\sqrt{2}} \log_3 81 = 1$

7. If log x + log y = log (x + y) then express x in terms of y. [Ans. $y/(y-1)$]

$\sqrt[3]{(0}$

2.5 LAWS OF INDICES

The word indices is a plural part of the word index (Power).
When we write

a^5, a is called the base and 5 is called the index of the base.

When we write .

$a^m x a^n$ then m and n are called indices.

Now $a^2 = a \times a, a^3 = a \times a \times a, a^4 = a \times a \times a \times a$

i.e., the index of the base indicates the number of times the base should be multiplied .However ,if index is a fraction it indicates the root,e.g.,

$a^{1/2} = \sqrt{a}, a^{1/3} = \sqrt[3]{a}, a^{2/3} = \sqrt[3]{a^2}$

The Radical sign $\sqrt{\ }$ with the number indicates the root,e.g. $\sqrt[3]{\ }$ indicates cube root and $\sqrt[4]{\ }$ indicates fourth root. The Radical sign without number indicate square root.

Quantities like $a^{m/n}$ written as $\sqrt[n]{a^m}$ are called radicals. The term under the radical sign is called the radicand and the number with the radical sign $(n \text{ in } \sqrt[n]{a^m})$ is called the index of the radical.

It is not possible to determine exactly every root of every positive number.

For example, $\sqrt{2}, \sqrt{3}, \sqrt[2]{5}$ etc. can not be determined exactly, and such quantities are called Surds or irrational quantities.

Surds cannot be expressed as the ratio of two integers and their values are usually calculated with the help of logarithms,

(i) $a^m \times a^n = a^{m+n}$

(ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$

where m and n are positive or negative, integral or fractional and a is a non zero real number.

Law I If m and n are positive integars then

$a^m x a^n = a \text{ x } a \text{ x } a \text{ x } a \text{ ..... m factors}$

$\times a \times a \times a........n$ factors

$= a \times a \times a \times a........(m+n)$ factors a^{m+n}

Law II $\dfrac{a^m}{a^n} = a^{m-n}$

Law III $(a^m)^n = a^m \times a^m \times a^m \times a^m \times .....n \text{ factors}$

$= a^{(m+m+m+......n \text{ terms})} = a^{mn}$

We have seen that

(1) $a^m = a \times a \times a \times ....m \text{ factors}$ for any non zero real number a and any positive integar m

(2) $a^0 = 1$ for any non-zero real number a.

(3) $a^{-k} = \frac{1}{a^k}$ for any non zero real number a and any integar k.

For any positive real number a and any positive integar m,the mth root of a is defined and denoted by $a^{1/m}$ or $\sqrt[m]{a}\left[a^{1/m} = \sqrt[m]{a}\right]$ and is a real number .But when m is an even positive integar and a is a negative real number .then $a^{1/m}$ is not a real number.

e.g, $\left(-\frac{1}{2}\right)^{\frac{1}{5}}, (-3)^{\frac{1}{4}}$ etc. are not real numbers.

In general $a^{m/n} = \sqrt[n]{a^m} = (a^m)^{\frac{1}{n}}$

Now $(a^{m/n})^n = a^{m/n} \times a^{m/n} \times a^{m/n}$ .... n factors

$= a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}.......n\ terms}$

$= a^{mn/n} = a^m$

Taking nth root of both sides, we get

$= a^{m/n} = \sqrt[n]{a^m}$

Note 1. If in an equation base on both sides is the same,powers can be equated, i e. if $x^a = x^b$ then $a = b$

SOLVED EXAMPLES

Example 43 : Show that the expression $\frac{2^{m+2}.3^{2m-n}.5^{m+n+2}.6^n}{6^m.10^{n+2}.15^m}$ is indepedent of m and n.

Solution : L.H.S.$=\frac{2^{m+2}.3^{2m-n}.5^{m+n+2}.6^n}{6^m.10^{n+2}.15^m}$

$= 2^{m+2}.2^{-m}.3^{-m}.3^{2m-n}.2^{-(n+2)}.5^{-(n+2)}.5^{m+n+2}.3^{-m}.5^{-m}.2^n.3^n$

$\because \frac{1}{6^m} = 2^{-m}.3^{-m}. \frac{1}{10^{n+2}} = 2^{-(n+2)}.5^{-(n+2)}, \frac{1}{15^m} = 3^{-m}.5^{-m}$

$\therefore$ L.H.S. $= 2^{m+2-m-n-2+n}.3^{-m+2m-n-m+n}.5^{-n-2+m+n+2-m}$

$= 2^o \times 3^o x5^o = 1$ which is independant of m and n.

Example 44: SImplify $\left[\frac{2^{1/3}.8^{2/3}.6^{-5/4}.3^{-3/4}}{9^{-1}\sqrt[3]{16}}\right]^{-4}$

Solution:

The given expression

$= \frac{2^{1/3}.(2^3)^{2/3}.(2 \times 3)^{-5/4}.3^{-3/4}.}{(3^2)^{-1}.(2^4)^{1/3}}^{-4}$

$= 2^{1/3}.2^{2}.2^{-5/4}.3^{-5/4}.3^{-3/4}.3^{2}.(2^{4})^{-1/3}\ ^{-4}$

$= \left[2^{(1/3)+2-(5/4)-4/3}(3^{-5/4-(3/4)+2})\right]^{-4}$

$= \left[2^{-1/4}.3^{0}\right]^{-4} = \left[2^{-1/4}\right]^{-4} = 2$

Example 45: Simplify $\dfrac{(3^{2n}-5\times 3^{2n-2})(5^{n}-3\times 5^{n-2})}{5^{n-4}[9^{n+3}-3^{2n}]}$

Solution : The Given Expression

$= \dfrac{(3^{2n}-5\times 3^{2n}x\,3^{-2})(5^{n}-3\times 5^{n}\times 5^{-2})}{5^{n}\times 5^{-4}[3^{2n}x\,3^{6}-3^{2n}]}$

$= \dfrac{3^{2n}\left(1-5/9\right)5^{n}\left(1-3/25\right)}{5^{n}x3^{2n}x\dfrac{1}{5^{4}}[3^{6}-1]} \quad = \dfrac{4}{9}x\dfrac{22}{25}x\dfrac{625}{728} = \dfrac{275}{819}$

Example 46 : Simplify $\dfrac{4^{n}\times 20^{m-1}\times 12^{m-1}\times 15^{m+n-2}}{16^{m}\times 5^{2m+n}\times 9^{m-1}}$

Solution : The Given Expression

$= \dfrac{4^{n}\times(4\times 5)^{m-1}\times(4\times 3)^{m-n}\times(3\times 5)^{m+n-2}}{4^{2m}\times 5^{2m+n}\times 3^{2m-2}}$

$= 4^{n+m-1+m-n-2m}\times 5^{m-1+m+n-2-2m-n}\times 3^{m-n+m+n-2-2m+2}$

$= 4^{-1}\times 5^{-3} = \dfrac{1}{4}\times\dfrac{1}{125} = \dfrac{1}{500}$

Example 47 : if $a^{x}=b^{y}=c^{z}$ and $b^{2}=ac$ prove that.

$\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{2}{y}$

Solution : Let $a^{x}=b^{y}=c^{z}=k$

$\therefore \quad a=(k)^{\frac{1}{x}}, b=(k)^{\frac{1}{y}}, c=(k)^{\frac{1}{z}}$

As per the equation,

$b^{2}=ac \Rightarrow (k)^{\frac{2}{y}} = k^{\frac{1}{x}}.k^{\frac{1}{z}} = k^{\frac{1}{x}+\frac{1}{z}}$

Equating powers on the same base

$\dfrac{2}{y}=\dfrac{1}{x}+\dfrac{1}{z}$

Example 48 : Simplify $\dfrac{(x^{a+b})^2.(x^{b+c})^2.(x^{c+a})^2}{(x^a,x^b,x^c)^4}$

Solution : $=\dfrac{(x^{a+b})^2.(x^{b+c})^2.(x^{c+a})^2}{(x^a,x^b,x^c)^4}$

$=\dfrac{x^{2a+2b}.x^{2b+2c}.x^{2c+2a}}{(x^a,x^b,x^c)^4}$ (since $(a^m)^n = a^{m/n}$)

$=\dfrac{x^{2a+2b+2b+2c+2c+2a}}{(x^{a+b+c})^4}$ $(a^m \times a^n = a^{m+n})$

$=\dfrac{x^{4a+4b+4c}}{x^{4a+4b+4c}}=1$

Example 49 : Find the value of

$$\frac{x^{4/7}x^{3/5}\sqrt[7]{x^3}}{\sqrt[8]{x^{-3}}\sqrt[5]{y^5x^3}}\times\frac{y^2}{(x^{1/8})^3}$$

Solution : Now $=\dfrac{x^{4/7}x^{3/5}\sqrt[7]{x^3}}{\sqrt[8]{x^{-3}}\ \sqrt[5]{y^5x^3}}\times\dfrac{y^2}{\left(x^{1/8}\right)^3}$

$$=\frac{x^{4/7}x^{3/5}x^{3/7}}{x^{-3/8}y^{5/5}x^{3/5}}\times\frac{y^2}{x^{3/8}}=\frac{x^{4/7+3/5+3/7}}{x^{3/5-3/8}.y}\times\frac{y^2}{x^{3/8}}$$

$=[x^{4/7+3/5+3/7-3/5+3/8-3/8}][y^{2-1}]=xy.$

Example 50 : If $m=a^x, n=a^y$ and $a^2=[m^yn^x]^z$ Prove that xyz = 1

Solution :

$m = a^x \quad m^y \quad a^{xy}, n \quad a^y \quad n^x \quad a^{xy}$

$[m^yn^x]^z = [a^{xy}a^{xy}]^z = a^{2xyz} = a^2$ (given) $\quad xyz \quad 1$

Example 51 : If $x^a = y, y^b = z, z^c = x$ prove that abc = 1.

Solution : We are given that

$x^a = y, \ldots.(i)$ $y^b = z, \ldots.(ii)$ $z^c = x, \ldots.(iii)$

substituting the value of y from (i) in (ii)

$(x^a)^b = z$ i.e., $x^{ab} = z$ ....(iv)

substituting the value of x from (iii) in (iv)

$(z^c)^{ab} = z$ i.e. $z^{abc} = z$

Equating powers on the same base we get abc = 1

abc = 1

Example 52 : Simplify $\sqrt{a\sqrt{a\sqrt{a\sqrt{a}}}}$ for $a = 3^{16/15}$

Solution : (Note carefully)

$$\sqrt{a\sqrt{a\sqrt{a\sqrt{a}}}} = a^{1/2}.(a^{1/2})^{1/2}\left\{(a^{1/2})^{1/2}\right\}^{1/2} \left\{(a^{1/2})^{1/2}\right\}^{1/2 \ 1/2}$$

$$\Rightarrow a^{1/2}.a^{1/4}.a^{1/8}.a^{1/16} = a^{15/16}$$

when $a = 3^{16/15}$, then $\sqrt{a\sqrt{a\sqrt{a\sqrt{a}}}} = (3^{16/15})^{15/16} = 3$

Example 53: If $a = 2^{1/3} - 2^{-1/3}$ show that $2a^3 + 6a - 3 = 0$

Solution : Given that $a = 2^{1/3} - 2^{-1/3}$

Taking cube of both sides

$$a^3 = 2 - 2^{-1} - 3(2^{1/3} - 2^{-1/3}) = 2 - \frac{1}{2} - 3a = \frac{3}{2} - 3a$$

i.e., $2a^3 + 6a - 3 = 0$

Example 54: if $x = 5 - 5^{2/3} - 5^{1/8}$, prove that

$x^3 - 15x^2 + 60x - 20 = 0$

Solution : $(5 - x) = 5^{2/3} + 5^{1/3}$

Cube both sides and simplify,

Example 55: if $a^b = b^a$ show that $\left(\frac{a}{b}\right)^{a/b} = a^{(a/b)-1}$ and if a=2b show that b=2

Solution : $a^b = b^a \Rightarrow b = a^{b/a}$

$$\frac{a}{b}^{a/b} = \frac{a}{a^{b/a}}^{a/b} = \frac{a^{a/b}}{a} = a^{(a/b)-1}$$

If a=2b then $\left(\frac{a}{b}\right)^{a/b} = a^{(a/b)-1} \Rightarrow \left(\frac{2b}{b}\right)^{2b/b} = (2b)^{(2b/b)-1}$

$\Rightarrow 4 = 2b \Rightarrow b = 2$

Example 56: if $(1.234)^a = (0.1234)^b = 10^c$, show that $\frac{1}{a} - \frac{1}{c} = \frac{1}{b}$

Solution:

$(1.234)^a = 10^c$ $\quad (0.1234 \quad 10)^a \quad 10^c$

or, $(0.1234)^a \times 10^a = 10^c$

or, $(0.1234)^a = 10^{c-a}$

or, $(0.1234) = 10^{\frac{c-a}{a}}$ .... (i)

and $(1.234)^b = 10^c \quad 0.1234 \quad 10^{c/b}$ .... (ii)

From (I) and (II), we get

$$10^{\frac{c-a}{a}} = 10^{\frac{c}{b}} \Rightarrow \frac{c-a}{a} = \frac{c}{b} \Rightarrow bc - ab = ac$$

$$\Rightarrow \frac{1}{a} - \frac{1}{c} = \frac{1}{b}$$

Example 57: Find the simplest value of

$$1-1\left\{1-\left(1-x^3\right)^{-1}\right\}^{-1} \quad ^{-1/3} \quad \text{when } x = 0, x = 0.1$$

Solution :

$$= \left[1-1\left\{1-\frac{1}{1-x^3}\right\}^{-1}\right]^{\frac{-1}{3}}$$

$$= \left[1-1\left\{\frac{1-x^3-1}{1-x^3}\right\}^{-1}\right]^{\frac{-1}{3}}$$

$$= \left[1-1\left\{\frac{-x^3}{1-x^3}\right\}^{-1}\right]^{\frac{-1}{3}}$$

$$= \quad 1-1 \quad \frac{1-x^3}{-x^3} \quad ^{\frac{-1}{3}}$$

$$= \left[1+\frac{1-x^3}{x^3}\right]^{\frac{-1}{3}} = \left[\frac{x^3+1-x^3}{x^3}\right]^{\frac{-1}{3}}$$

$$= \left[\frac{1}{x^3}\right]^{\frac{-1}{3}} = \left[x^3\right]^{\frac{1}{3}} = x = 0 \quad \text{when} \quad x = 0$$

$$= 0.1 \text{ when } x = 0.1$$

2.6 PERMUTATION AND COMBINATION

2.6.1. PERMUTATION :

Definition :

The different arrangements which can be made out of a given set of things, by taking some or all of them at a time are called *permutations*.

Thus the permutations of three letters a, b, c taking one, two or three at a time are respectively :

one :	a	b	c			
two :	ab	bc	ca	ba	cb	ac
three :	abc	bca	cab	acb	bac	cba

The number of permutations of *n* different things, taken r at a time, usually symbolised by ^nP_r or $_nP_r$.

Thus the number of arrangements (or Permutations) of 3 things taken 1, 2 and 3 at a time are respectively : $^3P_1$, $^3P_2$ and $^3P_3$.

General Principle :

If one operation can be performed in m different ways and corresponding to any one of such operations if a second operation can be performed in n different ways, then the total number of performing the two operations is $m \times n$.

The above principle is applied in the following theory of permutations.

Permutations of things all different :

To *find the number of permutations of n different things taken r* ($r \leq n$) *at a time.*

This is the same thing of finding out the number of different ways in which r places can be filled up by the *n* things taking one in each place.

The first place can be filled up in *n* ways since any one of the *n* different things can be put in it.

When the first place has been filled up in any of these *n* ways, the second place can be filled up in $(n - 1)$ ways, since any one of the remaining $(n - 1)$ things can be put in it.

Now corresponding to each of filling up the first place, there are $(n - 1)$ ways of filling up the second, the first two places can be filled up in $n (n - 1)$ ways.

Again, when the first two places are filled up in any one of the $n (n - 1)$ ways, the third place can be filled up by $(n - 2)$ ways, for there are now $(n - 2)$ things, at our disposal, to fill up the third place, Now corresponding to the each way of filling up the first two places, there are clearly $(n - 2)$ ways of filling up the third place. Hence the first three places can be up in $n (n -1) (n - 2)$ ways.

Proceeding similarly and noticing that the number of factors at any stage, is always equal to the number of places to be filled up, we conclude that the total number of ways in which r places can be filled up.

$$= n (n - 1) (n - 2) \ldots\ldots \text{ to r factors}$$

$$= n (n - 1) (n - 2) \ldots\ldots\ldots \{n - (r - 1)\}$$

$$= n (n - 1) (n - 2) \ldots\ldots (n - r + 1)$$

Hence, using the symbol of permutation, we get,

$$^nP_r = n (n - 1) (n - 2) \ldots (n - r + 1)$$

Cor. The number of permutation of n different things taking all at a time is given by

^nP_n = n (n – 1) (n – 2) ...... to *n* factors

= n (n – 1) (n – 2) ..... 3 × 2 × 1 [putting r =n]

Again, $^nP_{n-1}$ = n (n – 1) (n – 2) .... 3 × 2 [putting r = n – 1]

$\therefore\ ^nP_n = {}^nP_{n-1}$

Factorial notation :

The continued product of first *n* natural numbers, i.e., 1, 2, 3, ... (*n – 1*) *n*, is generally denoted by the symbol $\underline{|n}$ or *n*! which is read as 'factorial n".

Thus, 5! = 1× 2 × 3 × 4 × 5 =120 (= 5 × 4!) = (5 × 4 × 3!)

7! = 1 × 2 × 3 × 4 × 5 × 6 × 7 = (1 × 2 × 3 × 4 × 5 × 6) × 7 = 7 × 6!

n! = 1 × 2 × 3 ×.....× (n – 2) × (n – 1) n = ^nP_n ; O! = 1

Obs. ^nP_r = n (n – 1) (n – 2) ... (n – r + 1)

$$= \frac{n(n-1)(n-2).....(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Permutation of things when they are not all different :

To find the number of permutation of n things taken all together, the things are not all different.

Suppose that *n* things be represented by *n* letters and *p* of them be equal to a, q of them be equal to b and r of them be equal to c and the rest be all different. Let X be the required number of permutations.

$$\therefore\ X = \frac{(n)!}{(p)!(q)!(r)!}$$

Cor. The above method is also applicable when more than three letters are repeated.

Example 58 : Word PURPOSE can be arranged all together in $\frac{7!}{2!}$ as 2P's are there.

Permutations of things which may be repeated :

If n different things are taken r at a time, in which any item can be repeated without any restriction, the total number of possible arrangements is n^r.

Permutations in a ring or in a circle :

When things are arranged in a row, we find two ends in each arrangement, while when the things are arranged in circle, there is no such end.

Thus the number of ways in which *n* different things can be arranged in a circle taking all together is (*n – 1*) !, since any one of the things placed first is fixed and remaining (*n – 1*) things can now be arranged in (*n –1*) ! ways.

Example 59 : 21 boys can form a ring in (21 – 1) ! = 20! ways, If, again the distinction between the clockwise and counter-clockwise arrangements is not made, then the number of ways is ½ (n – 1) !.

Example 60: 10 different beads can be placed in a necklace in $\frac{1}{2}\times(10-1)! = \frac{1}{2}\times 9!$ ways.

Restricted Permutation :

(i) The number of permutations of *n* different things taken *r* at a time in which *p* particular things never occur is $^{n-p}P_r$.

Keeping aside the *p* particular things, fill up the *r* places with the remaining *n – p* things.

Hence, number of ways = $^{n-p}P_r$.

Example 61 : In how many of the permutations of 8 things taken 3 at a time, will two particular things never occur ?

Solution :

Here, n = 8, r = 3, p = 2,

Hence, Number of ways = $^{n-p}P_r = {}^{8-2}P_3 = {}^6P_3 = 120$.

(ii) The number of permutations of *n* different things taken *r* at a time in which *p* particular things are always present is $^{n-p}P_{r-p} \times {}^rP_p$.

Example 62 : In how many of the permutations of 8 things taken 3 at a time, will two particular things always occur ?

Solution :

Here, n = 8, r = 3, p = 2,

Number of ways $= {}^{n-p}P_{r-p} \times {}^rP_p = {}^{8-2}P_{3-2} \times {}^3P_2 = {}^6P_1 \times {}^3P_2 = 6 \times 3 = 18$.

SOLVED EXAMPLES

Example 63 : Find the values of– (i) $^7P_5$ (ii) $^7P_1$ (iii) $^7P_0$ (iv) $^7P_7$

Solution :

(i) $$^7P_5 = \frac{7!}{(7-5)!} = \frac{7!}{2!} = \frac{7.6.5.4.3.2.1}{2.1} = 7.6.5.4.3 = 2520$$

(ii) $$^7P_1 = \frac{7!}{(7-1)!} = \frac{7!}{6!} = 7 \times \frac{6!}{6!} = 7$$

(iii) $$^7P_0 = \frac{7!}{(7-0)!} = \frac{7!}{7!} = 1$$

(iv) $$^7P_7 = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1!} = 7.6.5.4.3.2.1 = 5040$$

Example 64 : If $^nP_2 = 110$, *n*.

Solution :

$$^nP_2 = \frac{n!}{(n-2)!} = 110 \text{ or, } \frac{n(n-1)(n-2)!}{(n-2)!} = 110$$

or, $n(n-1) = 110 = 11 \times 10 = 11 \times (11-1) \quad \therefore \; n = 11$

Example 65 : Solve for *n* given $^nP_4 = 30 \times {}^nP_2$

Solution :

$$^nP_4 = 30 \times {}^nP_2 \text{ or, } \frac{n!}{(n-4)!} = 30 \times \frac{n!}{(n-2)!}$$

or, $\frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!} = 30 \times \frac{n(n-1)(n-2)!}{(n-2)!}$

or, n(n – 1) (n – 2) (n – 3) = 30 × n(n – 1) or, (n – 2) (n – 3) = 30

or, $n^2 - 5n - 24 = 0$ or, (n – 8) (n + 3) = 0

or, n = 8, – 3 (inadmissible)

Example 66 : Solve for *n* given $\frac{^nP_5}{^nP_3} = \frac{2}{1}$

Solution :

$\frac{^nP_5}{^nP_3} = \frac{2}{1}$ or, $^nP_5 = 2 \times {^nP_3}$

or, $\frac{n!}{(n-5)!} = 2 \times \frac{n!}{(n-3)!}$ or, $1 = 2 \times \frac{1}{(n-3)(n-4)}$

or, $n^2 - 7n + 10 = 0$ or, n = 5, 2 (inadmissible).

Example 67: In how many ways 6 books out of 10 different books can be arranged in a book-self so that 3 particular books are always together?

Solution :

At first 3 particular books are kept outside. Now remaining 3 books out of remaining 7 books can be arranged in $^7P_3$ ways. In between these three books there are 2 places and at the two ends there are 2 places i.e. total 4 places where 3 particular books can be placed in $^4P_1$ ways. Again 3 particular books can also be arranged among themselves in 3! ways.

Hence, required no. of ways $= {^7P_3} \times {^4P_1} \times 3! = \frac{7!}{4!} \times \frac{4!}{3!} \times 3!$ = 7.6.5.4.3.2.1 = 5040.

Example 68: In how many ways can be letters of the word TABLE be arranged so that the vowels are always (i) together (ii) separated ?

Solution :

(i) In the word there are 2 vowels, 3 consonants all different. Taking the 2 vowels (A, E) as one letter we are to arrange 4 letters (i.e. 3 consonants + 1) which can be done in 4 ! ways. Again 2 vowels can be arranged among themselves in 2 ! ways.

Hence, required number of ways = 4! × 2! = 48.

(ii) Without any restriction (i.e. whether the vowels, consonants are together or not) all the different 5 letters can be arranged in 5! ways. Arrangement of vowels together is 48 (shown above)

Hence, Required number of ways = 5! – 48 = 120 – 48 = 72.

Example 69 : Find the how many ways can be letters of the PURPOSE be rearranged–

(i) keeping the positions of the vowels fixed ;

(ii) without changing the relative order to the vowels and consonants.

Solution :

(i) In the word, there are 3 vowels and 4 consonants. Since the positions of all vowels fixed, we are to rearrange only 4 consonants, in which there 2 P, so the arrangement is

$$\frac{4!}{2!}=\frac{4\times3\times2!}{2!}=4\times3=12$$

(ii) The relative order of vowels and consonants unaltered means that vowel will take place of vowel and consonant will take place of consonant. Now the 3 vowels can be arranged among themselves in 3! ways, while 4 consonants with 2P can be arranged in

$$\frac{4!}{2!}=\frac{4\times3\times2!}{2!}=4\times3=12 \text{ ways.}$$

So total number of ways of rearrangement in which the given arrangement is included = 3! × 12 = 6 × 12 = 72

Hence, Required number of arrangement = 72 – 1 = 71.

Example 70 : How many numbers between 5000 and 6000 can be formed with the digits 3, 4, 5, 6, 7, 8?

Solution :

The number to be formed will be of 4 figures, further digit 5 is to be placed in 1st place (from left). Now the remaining 3 places can be filled up by the remaining 5 digits in ${}^5P_3$ ways.

Hence, required no. $={}^5P_3\times1=\frac{5!}{2!}=60$

Example 71 : In how many ways can be letters of the word SUNDAY be arranged? How many of them do not begin with S? How many of them do not begin with S, but end with Y?

There are 6 letters in the word SUNDAY, which can be arranged in 6! = 720 ways.

Now placing S in first position fixed, the other 5 letters can be arrange in (5)! = 120 ways.

The arrangements of letters that do not begin with S = (6) ! – (5) ! = 720 – 120 = 600 ways.

Lastly, placing Y in the last position, we can arrange in (5) ! = 120 ways and keeping Y in the last position and S in the first position, we can arrange in (4) ! = 24 ways.

Hence, the required no. of arrangements = (5) ! – 4 ! = 120 – 24 = 96 ways.

(Problems regarding ring or circle)

Example 72 : In how many ways 8 boys can form a ring?

Solution :

Keeping one boy fixed in any position, remaining 7 boys can be arranged in 7 ! ways.

Hence, the required on. of ways = 7 ! = 7. 6. 5. 4. 3. 2. 1 = 5040.

Example 73: In how many ways 8 different beads can be placed in necklace?

Solution :

8 beads can be arranged in 7 ! ways. In this 7 ! ways, arrangements counting from clockwise and anticlockwise are taken different. But necklace obtained by clockwise permutation will be same as that obtained from anticlockwise. So total arrangement will be half of 7 !.

Hence, required no. of ways = ½ × 7 ! = ½ × 5040 = 2520.

Example 74: In how many ways 5 boys and 5 girls can take their seats in a round table, so that no two girls will sit side by side.

Solution :

If one boy takes his seat anywhere in a round table, then remaining 4 boys can take seats in 4 ! = 24 ways. In each of these 24 ways, between 5 boys, if 5 girls take their seats then no two girls will be side by side. So in this way 5 girls may be placed in 5 places in 5 ! = 120 ways.

Again the first boy while taking seat, may take any one of the 10 seats, i.e., he may take his seat in 10 ways.

Hence, reqd. number ways = $24 \times 120 \times 10 = 28{,}800$.

SELF EXAMINATION QUESTIONS

1. Find the value of : (i) ${}^{10}P_2$ (ii) ${}^{10}P_0$ (iii) ${}^{10}P_{10}$ [Ans. (i) 90 (ii) 1 (iii) 10 !]

2. Find the value of n : ${}^{n}P_4 = 10$ ´ ${}^{n-1}P_3$ [Ans. 10]

3. Find the value of r : (i) ${}^{11}P_{3r} = 110$ (ii) ${}^{7}P_r = 2520$ [Ans. (i) 2 (ii) 5]

4. Find n : (i) if ${}^{n}P_5 : {}^{n}P_3 = 2 : 1$ (ii) ${}^{n}P_3$ ´ ${}^{n+2}P_3 = 5 : 12$ [Ans. (i) 5 (ii) 7]

5. Prove that "CALCUTTA" is twice of "AMERICA" in respect of number of arrangements of letters.

OBJECTIVE QUESTIONS

1. There are 20 stations on a railway line. How many different kinds of single first-class tickets must be printed so as to enable a passenger to go from one station to another? [Ans. 380]

2. Four travellers arrive in a town where there are six hotels. In how many ways can they take their quarters each at a different hotel? [Ans. 360]

3. In how many ways can 8 mangoes of different sizes be distributed amongst 8 boys of different ages so that the largest one is always given to the youngest boy? [Ans. 5040]

4. Find the number of different number of 4 digits that can be formed with the digits 1, 2, 3, 4, 5, 6, 7 ; the digits in any number being all different and the digit in the unit place being always 7. [Ans. 120]

5. How many different odd numbers of 4 digits can be formed with the digits 1, 2, 3, 4, 5, 6, 7 ; the digits in any number being all different? [Ans. 480]

6. How many number lying between 1000 and 2000 can be formed from the digits 1, 2, 4, 7, 8, 9 ; each digit not occurring more than once in the number? [Ans. 60]

7. Find the number of arrangements that can be made out of the letters of the following words :

(a) COLLEGE

(b) MATHEMATICS [Ans. (a) 1260 ; (b) 49, 89,600]

8. In how many ways can the colours of a rainbow be arranged, so that the red and the blue colours are always together? [Ans. 1440]

9. In how many ways 3 boys and 5 girls be arranged in a row so that all the 3 boys are together?

[Ans. 4320]

10. Find how many words can be formed of the letters in the word FAILURE so that the four vowels come together. [Ans. 576]

11. In how many ways can 7 papers be arranged so that the best and the worst papers never come together? [Ans. 3600]

12. In how many ways can the colours of the rainbow be arranged so that red and blue colours are always separated? [Ans. 3600]

13. Show that the number of ways in which 16 different books can be arranged on a shelf so that two particular books shall not be together is 14 (15) !

14. In how many ways can the letters of the word MONDAY be arranged? How many of them begin with M? How many of them do not begin with M but end with Y? [Ans. 720, 120, 96]

15. In how many ways can 5 boys form a ring? [Ans. 24]

16. In how many ways 5 different beads be strung on a necklace? [Ans. 12]

2.6.2. COMBINATION

Definition :

The different *groups* or collection or *selections* that can be made of a given set of things by taking some or all of them at a time, without any regard to the order of their arrangements are called their combinations.

Thus the combinations of the letters a, b, c, taking one, two or three at a time are respectively.

a	ab	abc
b	bc	
c	ca	

Combinations of things all different :

To find the number of combinations of *n* different things taken *r* ($r \le n$) at a time, i.e., to find the value of ^nC_r.

Let X denote the required number of combinations, i.e., X = ^nC_r.

Now each combination contains *r* different things which can be arranged among themselves in r ! ways. So X combinations will produce X. *r* ! which again is exactly equal to the number of permutations of *n* different things taken *r* at a time, i.e., ^nP_r

Hence, $X \times r! = {}^nP_r$

$$X = \frac{^nP_r}{r!} = \frac{n!}{r!(n-r)!} \qquad \text{Since, } {}^nP_r = \frac{n!}{(n-r)!}$$

$$\therefore {}^nC_r = \frac{n!}{r!(n-r)!}$$

Cor. $^nC_1 = n$ taking r = 1

$^nC_n = 1$, taking r = n

$^nC_0 = 1$, taking r = 0

Restricted Combination :

To find the number of combinations of n different things taken r at a time, with the following restrictions :

(i) p particular things always occur ; and

(ii) p particular things never occur.

(i) Let us first consider that p particular things be taken always ; thus we have to select (r – p) things from (n – p), which can be done in $^{(n-p)}C_{(r-p)}$ ways.

(ii) In this case, let those p things be rejected first, then we have to select *r* things from the remaining (n – p) things, which can be done in $^{n-p}C_r$ ways.

Total number of combinations :

To find the total number of combination of n different things taken 1, 2, 3 …. n at a time.

$= {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n$

Note. ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots.. + {}^nC_n = 2^n - 1$

GROUPING :

(A) If, it is required to form two groups out of (m + n) things, $(m \neq n)$ so that one group consists of *m* things and the other of *n* things. Now formation of one group represents the formation of the other group automatically. Hence the number of ways m things can be selected from (m + n) things.

$$ {}^{m+n}C_m = \frac{(m+n)!}{m!\,(m+n-m)\,!} = \frac{(m+n)\,!}{m!\,n!} $$

Note 1. If m =n, the groups are equal and in this case the number of different ways of subdivision $= \frac{(2m)\,!}{m!\,m!} \times \frac{1}{1!}$ since two groups can be interchanged without getting a new subdivision.

Note 2. If 2m things be divided equally amongst 2 persons, then the number of ways $\frac{(2m)\,!}{m!\,m!}$.

(A) Now (m + n + p) things $(m\ ^1\ n\ ^1\ p)$, to be divided into three groups containing *m, n, p* things respectively. *m* things can be selected out of (*m* + *n* + *p*) things in ${}^{m+n+p}C_m$ ways, then *n* things out of remaining (*n* + *p*) things in ${}^{n+p}C_n$ ways and lastly *p* things out of remaining p things in ${}^pC_p$ i.e., one way.

Hence the required number of ways is ${}^{m+n+p}C_m \times {}^{n+p}C_n$

$$ = \frac{(m+n+p)}{m!(n+p)\,!} \times \frac{(n+p)\,!}{n!\,p!} = \frac{(m+n+p)\,!}{m!\,n!\,p!} $$

Note 1. If now m = n = p, the groups are equal and in this case, the different ways of subdivision $= \frac{(3m)\,!}{m!\,m!\,m!} \times \frac{1}{3!}$ since the three groups of subdivision can be arranged in 3 ! ways.

Note 2. If 3m things are divided equally amongst three persons, the number of ways $= \frac{(3m)\,!}{m!\,m!\,m!}$

SOLVED EXAMPLES

Example 75 : In how many ways can be College Football team of 11 players be selected from 16 players?

Solution :

The required number $= {}^{16}C_{11} = \frac{16\,!}{11!\,(16-11)\,!} = \frac{16!}{11!\,5\,!} = 4,368$

Example 76: From a company of 15 men, how many selections of 9 men can be made so as to exclude 3 particular men?

Solution :

Excluding 3 particular men in each case, we are to select 9 men out of (15 – 3) men. Hence the number of selection is equal to the number of combination of 12 men taken 9 at a time which is equal to

$$ = {}^{12}C_9 = \frac{12\,!}{9\,!\,3\,!} = 220. $$

Example 77: There are seven candidates for a post. In how many ways can a selection of four be made amongst them, so that :

(i) 2 persons whose qualifications are below par are excluded?

(ii) 2 persons with good qualifications are included?

Solution :

(i) Excluding 2 persons, we are to select 4 out of 5 (= 7 – 2) candidates.

Number of possible selections = $^5C_4 = 5$.

(ii) In this case, 2 persons are fixed, and we are to select only 2 persons out of (7–2), *i.e.* 5 candidates. Hence the required number of selection = $^5C_2 = 10$.

Committee from more than one group :

Example 78: In how many ways can a committee of 3 ladies and 4 gentlemen be appointed from a meeting consisting of 8 ladies and 7 gentlemen? What will be the number of ways if Mrs. X refuses to serve in a committee having Mr. Y as a member?

Solution :

1st part. 3 ladies can be selected from 8 ladies in $^8C_3 = \frac{8!}{3!\,5!} = 56$ ways and

4 gentlemen can be selected from 7 gentlemen in $^7C_4 = \frac{7!}{4!\,3!} = 35$ ways

Now, each way of selecting ladies can be associated with each way of selecting gentlemen.

Hence, the required no. of ways = $56 \times 35 = 1960$.

2nd part : If both Mrs. X and Mr. Y are members of the committee then we are to select 2 ladies and 3 gentlemen from 7 ladies and 6 gentlemen respectively. Now 2 ladies can be selected out of 7 ladies in $^7C_2$ ways, and 3 gentlemen can be selected out of 6 gentlemen in $^6C_3$ ways.

Since each way of selecting gentlemen can be associated with each way of selecting ladies.

Hence, No. of ways $= {}^7C_2 \times {}^6C_3 = \frac{7!}{2!\,5!} \times \frac{6!}{3!\,3!} = 420$

Hence, the required no. of different committees, not including Mrs. X and Mr. Y

= 1960 – 420 = 1540.

Example 89 : From 7 gentlemen and 4 ladies a committee of 5 is to be formed. In how many ways can this be done to include at least one lady? [C.U. 1984]

Possible ways of formation of a committee are :–

(i) 1 lady and 4 gentlemen (ii) 2 ladies and 3 gentlemen

(iii) 3 ladies and 2 gentlemen (iv) 4 ladies and 1 gentleman

For (i), 1 lady can be selected out of 4 ladies in $^4C_1$ ways and 4 gentlemen can be selected from 7 gentlemen in $^7C_4$ ways. Now each way of selecting lady can be associated with each way of selecting gentlemen. So 1 lady and 4 gentlemen can be selected in $^4C_1 \times {}^7C_4$ ways.

Similarly,

Case (ii) can be selected in $^4C_2 \times {}^7C_3$ ways

Case (iii) can be selected in $^4C_3 \times {}^7C_2$ ways

Case (iv) can be selected in ${}^4C_4 \times {}^7C_1$ ways

Hence the total number of selections, in each case of which at least one lady is included

$= {}^4C_1 \times {}^7C_4 + {}^4C_2 \times {}^7C_3 + {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1$

$= 4 \times 35 + 6 \times 35 + 4 \times 21 + 1 \times 7$

$= 140 + 210 + 84 + 7 = 441.$

Example 80: In how many ways can a boy invite one or more of 5 friends?

Solution :

The number of ways $= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 2^5 - 1 = 32 - 1 = 31.$

Example 81 : In a group of 13 workers contains 5 women, in how many ways can a subgroup of 10 workers be selected so as to include at least 6 men? [ICWA (F) Dec 2005]

Solution :

In the given group there are 8 (= 13 – 5) men and 5 women in all. Possible cases of forming the subgroup of 10 workers.

	men	women	selections	
(i)	6	4	${}^8C_6 \times {}^5C_4$	$= 28 \times 5 = 140$
(ii)	7	3	${}^8C_7 \times {}^5C_3$	$= 8 \times 10 = 80$
(iii)	8	2	${}^8C_8 \times {}^5C_2$	$= 1 \times 10 = 10$

$\therefore$ reqd. no of ways = 230.

Example 82 : In how many ways 15 things be divided into three groups of 4, 5, 6 things respectively.

Solution :

The first group can be selected in ${}^{15}C_4$ ways :

The second group can be selected in ${}^{(15-4)}C_5 = {}^{11}C_5$ ways ;

and lastly the third group in ${}^6C_6 = 1$ way.

Hence the total number of ways $= {}^{15}C_4 \times {}^{11}C_5$

$$= \frac{15!}{4!\,11!} \times \frac{11!}{5!6!} = \frac{15!}{4!\,5!\,6!}$$

Example 83 : A student is to answer 8 out of 10 questions on an examination :

(i) How many choice has he?

(ii) How many if he must answer the first three questions?

(iii) How many if he must answer at least four of the first five questions?

Solution :

(i) The 8 questions out of 10 questions may be answered in ${}^{10}C_8$

Now $^{10}C_8 = \frac{10!}{8!2!} = \frac{10 \times 9 \times (8)!}{8!\,2!} = 5 \times 9 = 45$ ways

(ii) The first 3 questions are to be answered. So there are remaining 5 (= 8 – 3) questions to be answered out of remaining 7 (= 10 – 3) questions which may be selected in ${}^7C_5$ ways.

Now, $^7C_5 = 7.6 = 42$ ways.

(ii) Here we have the following possible cases :

(a) 4 questions from first 5 questions (say, group A), then remaining 4 questions from the balance of 5 questions (say, group B).

(b) Again 5 questions from group A, and 3 questions from group B.

For (a), number of choice is $^5C_4 \times {^5C_4} = 5 \times 5 = 25$

For (b) number of ways is $^5C_5 \times {^5C_3} = 1 \times 10 = 10$.

Hence, Required no. of ways = 25 + 10 = 35.

Example 84 : Given *n* points in space, no three of which are collinear and no four coplanar, for what value of *n* will the number of straight lines be equal to the number of planes obtained by connecting these points?

Solution :

Since no three points, are collinear, the number of lines = number of ways in which 2 points can be selected out of *n* points

$$= {^nC_2} = \frac{n(n-1)}{2} \text{ lines}$$

Again since three non-collinear points define a space and no four of the points are coplaner ; the number of planes = number of ways in which 3 points can be selected out of n points.

$$= {^nC_3} = \frac{n(n-1)(n-2)}{6}$$

Now, we have $= \frac{n(n-1)}{2} = \frac{1}{6} n(n-1(n-2))$; or, 6 = 2 (n – 2) Hence, n = 5

SELF EXAMINATION QUESTIONS

1. In an examination paper, 10 questions are set. In how many different ways can you choose 6 questions to answer. If however no. 1 is made compulsory in how many ways can you select to answer 6 questions in all? [Ans. 210, 126]

2. Out of 16 men, in how many ways a group of 7 men may be selected so that :

(i) particular 4 men will not come,

(ii) particular 4 men will always come? [Ans. 792 ; 220]

3. Out of 9 Swarjists and 6 Ministerialists, how many different committees can be formed, each consisting of 6 Swarajists and Ministerialists? [Ans. 1680]

4. A person has got 15 acquaintances of whom 10 are relatives. In how many ways may be invite 9 guests so that 7 of them would be relatives? [Ans. 1200]

5. A question paper is divided in three groups A, B and C each of which contains 3 questions, each of 25 marks. One examinee is required to answer 4 questions taking at least one from each group. In how many ways he can choose the questions to answer 100 marks [[Ans. 81]

[**hints :** $({^3C_1} \times {^3C_1} \times {^3C_2}) + ({^3C_1} \times {^3C_2} \times {^3C_1}) + ({^3C_2} \times {^3C_1} \times {^3C_1})$ etc.]

6. Out of 5 ladies and 3 gentlemen, a committee of 6 is to be selected. In how many ways can this be done : (i) when there are 4 ladies, (ii) when there is a majority of ladies? [Ans. 15, 18]

7. A cricket team of 11 players is to be selected from two groups consisting of 6 and 8 players respectively. In how many ways can the selection be made on the supposition that the group of six shall contribute no fewer than 4 players? [Ans. 344]

8. There are 5 questions in group A, 5 in group B and 3 in C. In how many ways can you select 6 questions taking 3 from group A, 2 from group B, and 1 from group C. [Ans. 180]

9. A question paper is divided into three groups A, B, C which contain 4, 5 and 3 questions respectively. An examinee is required to answer 6 questions taking at least 2 from A, 2 from B, 1 from group C. In how many ways he can answer. [Ans. 480]

10. (i) n point are in space, no three of which are collinear. If the number of straight lines and triangles with the given points only as the vertices, obtained by joining them are equal, find the value of *n*. [Ans. 5]

(ii) How many different triangles can be formed by joining the angular points of a decagon? Find also the number of the diagonals of the decagon. [Ans. 120 ; 35]

11. In a meeting after every one had shaken hands with every one else, it was found that 66 handshakers were exchanged. How many members were present at the meeting? [Ans. 12]

12. A man has 3 friends. In how many ways can be invite one or more of them to dinner? [Ans. 63]

13. In how many ways can a person choose one or more of the four electrical appliances ; T.V., Refrigerator, Washing machine, Radiogram? [Ans. 15]

14. In how many way can 15 things be divided into three groups of 4, 5, 6 things respectively?

[Ans. $(15)!/4!\,5!\,6!$]

15. Out of 10 consonants and 5 vowels, how many different words can be formed each consisting 3 consonants and 2 vowels. [Ans. 144000]

[**Hints :** $^{10}C_3 \times {}^5C_2 \times 5\,!$ & etc. here 5 letters can again be arranged among themselves in 5 ! ways.]

OBJECTIVE QUESTIONS

1. If $^{n}P_3 = 2.{}^{n-1}P_3$, find n [Ans. 6]

2. If $^{n}P_4 = 12\ {}^{n}P_2$, find n [Ans. 6]

3. Find n if $^{n}C_{n-2} = 21$ [Ans. 7]

4. If $^{18}C_r = {}^{18}C_{r+2}$ find the value of $^{r}C_5$

[Ans. 56]

5. If $^{n}C_n = 1$ then show that 0 ! = 1

6. If $^{n}P_r = 210$, $^{n}{}_{Cr} = 35$ find r [Ans. 3]

7. If $^{n}p_r = 336$, $^{n}C_r = 56$, find n and r [Ans. 8, 3]

8. $^{2n}C_3 : {}^{n}C_2 = 44 : 3$, find n [Ans. 6]

9. Prove that $^{10}P_{10} \times {}^{22}C_{12} = {}^{22}P_{10}$

10. Simplify : $^{4}P_2 \div {}^{4}C_2$ [Ans. 2]

11. If x ^{1} y and $^{11}C_x = {}^{11}C_y$, find the value of (x + y) [Ans. 11]

12. If $^{n}P_2 = 56$ find n [Ans. 8]

13. If $^{r}C_{12} = {}^{r}C_8$ find $^{22}C_r$ [Ans. 231]

14. If $^{7}P_r = 2520$ find r [Ans. 5]

2.7 SIMULTANEOUS LINEAR EQUATIONS

Introduction :

A linear equation of two unknowns x and y, is of the form ax + by + c = 0, where a > 0, b > 0.

Two such equations : $a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

form two simultaneous linear equations in x and y.

Methods of Solution

There are two methods, such as–

(1) Method of elimination,

(2) Rule of cross-multiplication

Method of Elimination

Example 85 :

Solve 3x + 4y = 11 … (i) 5x – 2y = 1 … (ii)

Solution :

Multiplying eqn. (I) by 5 and eqn. (ii) by 3, we find

15x + 20y = 55 …. (iii)

15x – 6y = 3 ….. (iv)

Now subtracting eqn. (iv) from eqn. (iii), 26y = 52 or, y = 2.

Putting this value of y in eqn. (i), 3x + 4.2 = 11 or, 3x = 11 – 8 or, x = 1

∴ x = 1, y = 2.

Rule of Cross-multiplication :

If two relations amongst three unknowns are given, the ratios of the three unknowns can be obtained by the rule of cross-multiplication.

For example, let such two relations are given by the following equations :

$a_1x + b_1y + c_1z = 0$ …. (1)

$a_2x + b_2y + c_2z = 0$ …. (2)

The rule states

$$\frac{x}{b_1c_2 - c_1b_2} = \frac{y}{c_1a_2 - a_1c_2} = \frac{z}{a_1b_2 - b_1a_2}$$

Example 86 : Solve 3x + 4y = 11 …. (1) 5x – 2y = 1 …. (2)

Solution :

The expression can be written as

3x + 4y – 11 = 0, $a_1 = 3$, $b_1 = 4$, $c_1 = -11$

5x – 2y – 1 = 0, $a_2 = 5$, $b_2 = -2$, $c_2 = -1$

$$\therefore x = \frac{4.(-1)-(-11).(-2)}{3(-2)-4.5} = \frac{-4-22}{-6-20} = \frac{-26}{-26} = 1$$

$$y = \frac{-11(5)-3(-1)}{3(-2)-4.5} = \frac{-55+3}{-6-20} = \frac{-52}{-26} = 2.$$

SOLVED EXAMPLES

Example 87 : Solve $4x + 2y - 15 = 5y - 3x + 16$ … (1)

$5x - y - 30 = 4(y - x) + 11$ … (2)

Solution :

From (1), $4x + 2y - 15 - 5y + 3x - 16 = 0$

Or, $7x - 3y = 31$ ………………. (3)

From (2) (in the same way)

$9x - 5y = 41$ ……………… ….. (4)

Multiplying (3) by 5 and (4) by 3, we find

$35x - 15y = 155$

$27x - 15y = 123$

Substracting. $8x = 32$ or, $x = 4$

Now putting $x = 4$ in eqn. (3)

$7.4 - 3y = 31$ or, $-3y = 31 - 28 = 3$ or, $y = -1$.

Example 88 : Solve $\frac{x}{4} + \frac{y}{5} = 22$ …. (1)

$\frac{x}{5} + \frac{y}{4} = 23$ …. (2)

Solution :

From (1), $\frac{5x + 4y}{20} = 22$ or, $5x + 4y = 440$ … (3)

From (2), $\frac{4x + 5y}{20} = 23$ or, $4x + 5y = 460$ … (4)

Multiplying (3) by 4 and (4) by 5 and then substracting we get $-9y = -540$ or, $y = 60$.

Putting this value of y in (3), we find $x = 40$.

2.7.1. QUADRATIC EQUATION

An equation in which the highest power of x (unknown) is two is called an equation of second degree or quadratic.

Thus, $x^2 - 5x + 6 = 0$, $x^2 - 9x = 0$, $x^2 = 0$ are all quadratic equations.

$ax^2 + bx + c = 0$, $(a \neq 0)$ is a standard form of quadratic equation. If $b = 0$ equation is pure quadratic. If $b \neq 0$ the equation is adfected quadratic.

Methods of solution

There are two methods of solution as follows :

(a) by factorisation, and

(b) by completing the square.

In the case of (a), we are to break middle term and hence to form two factors.

Example 89 : $2x^2 - 7x + 6 = 0$

L.H.S. $2x^2 - 3x - 4x + 6 = x(2x - 3) - 2(2x - 3) = (2x - 3)(x - 2)$

Now, $(2x - 3)(x - 2) = 0$ is true if either $2x - 3$ or, $x - 2 = 0$

From $2x - 3 = 0$ we get $2x = 3$ or $x = 3/2$, and from $x - 2 = 0$ we get $x = 2$

Hence the values (or roots) are 3/2, 2.

By the. Second method (b) for $ax^2 + bx + c = 0$, we have

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, (the proof is not shown at present)

Example 90 : Solve $2x^2 - 7x + 6 = 0$ Here $a = 2, b = -7, c = 6$

$$\therefore x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4.2.6}}{2.2} = \frac{7 \pm \sqrt{49 - 48}}{4} = \frac{7 \pm 1}{4} = \frac{8}{4}, \frac{6}{4} = 2, \frac{3}{2}$$

Example 91 : Solve $\frac{9}{x^2 - 27} = \frac{25}{x^2 - 11}$

Cross multiplying, $25x^2 - 675 = 9x^2 - 99$, or, $16x^2 = 576$ or, $x^2 = 36$ or, $x = \pm 6$

Example 92 : Solve $x^2 - 7x + 12 = 0$

Solution :

This can be expressed as $x^2 - 3x - 4x + 12 = 0$ or, $x(x - 3) - 4(x - 3) = 0$

or, $(x - 3)(x - 4) = 0$ Hence, $x = 3, 4$.

Alternatively, $x = \frac{-7 \pm \sqrt{49 - 48}}{2} = \frac{7 \pm 1}{2}$. Hence, $x = 4, 3$. (Here, $a = 1, b = -7, c = 12$)

Equation reducible to Quadratics :

There are various types of equations, not quadratic in form, which can be reduced to quadratic forms by suitable transformation, as shown below :

Example 93 : Solve $x^4 - 10x^2 + 9 = 0$.

Solution :

Taking, $x^2 = u$, we get $u^2 - 10u + 9 = 0$

or, $(u - 9)(u - 1) = 0$; either $(u - 9) = 0$ or, $(u - 1) = 0$ Hence, $u = 9, 1$.

When $u = 9$, $x^2 = 9$ or, $x = \pm 3$

Again, $u = 1$, $x^2 = 1$ or, $x = \pm 1$ Hence, $x = \pm 3, \pm 1$.

Here the power of x is 4, so we get four values of x.

Example 94 : Solve : $(1 + x)^{1/3} + (1 - x)^{1/3} = 2^{1/3}$

Solution :

We get, $(1 + x) + (1 - x) + 3(1 + x)^{1/3}(1 - x)^{1/3}.[(1 + x)^{1/3} + (1 - x)^{1/3}] = 2$ (cubing sides).

or, $2 + 3(1 - x^2)^{1/3}.2^{1/3} = 2$ or, $3(1 - x^2)^{1/3}.2^{1/3} = 0$

or, $(1 - x^2)^{1/3} = 0$, as $3.2^{1/3} \neq 0$

or, $1 - x^2 = 0$ (cubing again)

or, $x^2 = 1$ $\therefore$ $x = \pm 1$

Example 95 : Solve $\frac{6-x}{x^2-4}=\frac{x}{x+2}+2$

Solution :
Multiplying by the L.C.M. of the denominators, we find :

$6 - x = x(x-2) + 2(x^2-4)$ or, $3x^2 - x - 14 = 0$

or, $(3x-7)(x+2) = 0$

$\therefore$ either $3x - 7 = 0$ or, $x + 2 = 0$

$\therefore x=\frac{7}{3}$ or, -2

Now $x = -2$ does not satisfy the equation, $x=\frac{7}{3}$ is the root of the equation.

Example 96: Solve $4^x - 3.2^{x+2} + 2^5 = 0$.

Solution :
Here, $2^{2x} - 3.2^x.2^2 + 32 = 0$

or, $(2^x)^2 - 12.2^x + 32 = 0$ or, $u^2 - 12u + 32 = 0$ (taking $u = 2^x$)

or, $(u-4)(u-8) = 0$

$\therefore$ either $(u-4) = 0$ or, $(u-8) = 0$ $\therefore u = 4, 8$

When $u = 4$, $2^x = 4 = 2^2$ $\therefore x = 2$

Again $u = 8$, $2^x = 8 = 2^3$ $\therefore x = 3$.

Extraneous Solutions :

In solving equations, the values of x so obtained may not necessarily be the solution of the original equation. Care should be taken to verify the roots in each case, and also square roots (unless stated otherwise) are to be taken as positive.

Example 97: Solve $x^2+7x+\sqrt{x^2+7x+9}=3$

Solution :

Adding 9 to both sides, we have $x^2+7x+9+\sqrt{x^2+7x+9}=12$

Now putting $u=\sqrt{x^2+7x+9}$, the equation reduces to

$u^2 + u - 12 = 0$
or, $u^2 + 4u - 3u - 12 = 0$ or, $u(u+4) - 3(u+4) = 0$
or, $(u-3)(u+4) = 0$ $\therefore u = 3, -4$.
Since u is not negative, we reject the value -4 for u.

When $u=3, \sqrt{x^2+7x+9}=3$ or, $x^2 + 7x + 9 = 9$

or, $x(x+7) = 0$ or, $x = 0, -7$.

Example 98 : Solve $(x^2 + 3x)^2 + 2(x^2 + 3x) = 24$

Solution :
Let $x^2 + 3x = u$, so that equ. becomes $u^2 + 2u = 24$ or, $u^2 + 2u - 24 = 0$
or, $u^2 + 6u - 4u - 24 = 0$ or, $u(u+6) - 4(u+6) = 0$

or, (u + 6) (u – 4) = 0 or, u = – 6, u = 4

For u = – 6, $x^2 + 3x = -6$ or, $x^2 + 3x + 6 = 0$

$x = \frac{-3 \pm \sqrt{9-24}}{2} = \frac{-3 \pm \sqrt{-15}}{2}$, rejected as values are not real

For u = 4, $x^2 + 3x = 4$ or, $x^2 + 3x - 4 = 0$

or (x + 4) (x – 1) = 0 or, x = – 4, 1.

SELF EXAMINATION QUESTIONS

1. $6x^2 - 11x - 10 = 0$. [Ans. $\frac{5}{2}, -\frac{2}{3}$]

2. $(2x-1)^{1/3} = (6x-5)^{1/3}$ [Ans. 1]

3. $x + \frac{1}{x} = \frac{10}{3}$. [Ans. 3, $\frac{1}{3}$]

4. $4x + \frac{4}{x} = 17$ [Ans. 4, $\frac{1}{4}$]

5. $\frac{1}{x+a+b} = \frac{1}{x} + \frac{1}{a} + \frac{1}{b}$ [Ans. – a, – b]

6. $\frac{x}{3} + \frac{3}{x} = 4\frac{1}{4}$. [Ans. 12, $\frac{3}{4}$]

7. $\frac{7x}{6} - \frac{30}{x} = \frac{x}{3}$. [Ans. ± 6]

8. $3^x + \frac{1}{3x} = \frac{10}{3}$. [Ans. ± 1]

9. $2x^{-1} + x^{-1/2} = 6$. [Ans. $\frac{1}{4}, \frac{4}{9}$]

10. $2^{x-2} + 2^{3-x} = 3$. [Ans. 2, 3]

11. $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$ [Ans. $\frac{9}{13}, \frac{4}{13}$]

2.8 MATRICES AND DETERMINANTS

2.8.1. Matrix

A set of mn numbers arranged in the form of rectangular array of m rows and n columns is called an (m X n) matrix (to be read as 'm' by'n'matrix)

An m x n matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}_{mxn}$$

In compact form the above matrix is represented by A = $[a_{ij}]_{mxn}$

Where a_{ij} denote *i*th row & *j*th column element.

Example 99 : Find A_{3x2} matrix where $a_{ij} = (i + j)^2$

Solution:

$$A_{3X2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \Rightarrow \begin{bmatrix} (1+1)^2 & (1+2)^2 \\ (2+1)^2 & (2+2)^2 \\ (3+1)^2 & (3+2)^2 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 9 & 16 \\ 16 & 25 \end{bmatrix}$$

Example 100: Find $A_{3 \times 3}$ where $a_{ij} = \begin{Bmatrix} i+j; & i < j \\ i-j; & i \geq j \end{Bmatrix}$

Solution:

$$A_{3X3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 1+2 & 1+3 \\ 2-1 & 2-2 & 2+3 \\ 3-1 & 3-2 & 3-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & 5 \\ 2 & 1 & 0 \end{bmatrix}$$

SELF EXAMINATION QUESTIONS

(1) Find A_{3x3} where $a_{ij} = \frac{i}{j}$ $\left(\text{Ans.} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix} \right)$

(2) Find A_{2x3} where $a_{ij} = i + 2j$ $\left(\text{Ans.} \begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix} \right)$

TYPE OF MATRICES

1. Row Matrix

A matrix having only one row and n columns is called row matrix. i.e. $[a_{11}\ a_{12}\ a_{13} \ldots\ a_{1n}]_{1xn}$
For example $[2\ 3\ 4\ 5]_{1x4}$

2. Column Matrix

A matrix having only m rows and one column is called column matrix i.e. $\begin{matrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{matrix}$

i.e. $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}_{4x1}$

3. Square Matrix

A matrix order m x n called square matrix where m = n, i.e., it has same numbers of rows and columns.

Say for example

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2x2} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3x3}$

4. Diagonal Matrix

A square matrix having main diagonal elements non zero and other elements are zero is called diagonal matrix.

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_{2x2'} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3x3}$

5. Scalar Matrix

A diagonal matrix having each diagonal element equal is called scalar matrix.
For i.e.

$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}_{2x2'} \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}_{3x3}$

6. Unit (or Identify) Matrix

A diagonal matrix having each diagonal element equal to one is called unit matrix. It denote by I_n. Thus,

$$I_2 = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, I_3 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

7. Null (or Zero) Matrix

A matrix of order m x n having all elements zero is called Null matrix and it's denoted by 0. i.e.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2x3}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3x1}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2x2}$$

8. Upper Triangular Matrix

A square is said to be an upper triangular matrix if all elements below the main diagonal are zero. As,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}_{3x3}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}_{4x4}$$

9. Lower Triangular Matrix

A square matrix is said to be an lower triangular matrix if all elements above the main diagonal are zero. i.e.

$$\begin{matrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{matrix}_{3x3}, \begin{matrix} a & 0 & 0 & 0 \\ b & e & 0 & 0 \\ c & f & h & 0 \\ d & g & i & j \end{matrix}_{4x4}$$

OPERATIONS ON MATRICES

Equality of Matrices

Two matrices are said to be equal if they are of same order and their corresponding elements are equal.

Say for Example of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

then a = 1, b = 2, c = 3, d = 4

Scalar multiplication of a matrix

If a scalar quantity (say k) multiply by a matrix A of order m x n, then k is multiply by each element of matrix A.

i.e. $kA = ka_{ij}$ where $A = a_{ij}$

Example

If A example of $=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then kA $=\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

Addition or subtraction of two matrices

The sum or difference of two matrices is defined only for the matrices of the same order. To add/subtract two matrices we add/subtract their corresponding elements.

Example 101 :

$$\begin{matrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{matrix} + \begin{matrix} 1 & 3 & 2 \\ -1 & 6 & 8 \end{matrix}$$

$$=\begin{bmatrix} 3 & 6 & 6 \\ 4 & 12 & 15 \end{bmatrix}$$

Example 102 :

$$\begin{matrix} 3 & 5 \\ 7 & 9 \end{matrix} - \begin{matrix} 2 & 6 \\ 8 & 2 \end{matrix}$$

$$=\begin{bmatrix} 1 & -1 \\ -1 & 7 \end{bmatrix}$$

Properties of matrix addition

- A + B = B + A (Commutative law)
- (A + B) + C = A + (B + C) (Associative law)
- k(A + B) = KA + KB (K is a Scalar)
- A + (–A) = (–A) + A = O
- A + O = O +A = A (O is null matrix)

SELF EXAMINATION QUESTIONS

(1) Evaluate

$$\begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix} + 2\begin{bmatrix} 1 & 3 \\ 4 & 9 \end{bmatrix} \qquad \left(\text{Ans.}\begin{bmatrix} 5 & 8 \\ 12 & 25 \end{bmatrix}\right)$$

(2) Evaluate

$$\begin{bmatrix} 3 & 5 \\ 2 & 7 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 9 \\ 8 & 15 \end{bmatrix} \qquad \left(\text{Ans.}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

Multiplication of two matrices

The Product AB of two matrices A and B is defined only if the number of columns in A is equal to the number of rows in B. If A is of order m x n and B is of order n x s. Then order of AB is m x s

To multiply A with B, elements of the ith row of A are to be multiplied by corresponding elements of jth column of B and then their sum is taken

i.e. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ & $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

Then $AB = \begin{matrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{matrix}$

Example 103 : Evaluate AB where $A = [1\ 3\ 2]_{1x3}$ $B = \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}_{3x1}$

Solution: $AB = [1 X 4 + 3 X 1 + 2 X 7]_{1x1}$

$= [21]$

Example 104 : Evaluate AB where $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}_{3x2}$ $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2x3}$

Solution: $AB = \begin{bmatrix} 2x1+3x2 & 2x3+3x4 & 2x5+3x6 \\ 4x1+5x2 & 4x3+5x4 & 4x5+5x6 \\ 6x1+7x2 & 6x3+7x4 & 6x5+7x6 \end{bmatrix}$

$= \begin{bmatrix} 8 & 18 & 28 \\ 14 & 32 & 50 \\ 20 & 46 & 72 \end{bmatrix}$

Example 105 : Evaluate AI where $A = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$

Solution: $AI = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3x1+2x0 & 3x0+2x1 \\ 4x1+7x0 & 4x0+7x1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$

Properties of Matrix Multiplication

(i) A(BC) = (AB) C (Associative law) (Where A_{mxn} B_{nxs} & C_{sxt})

(ii) A(B+C) = AB + AC (Distributive law)

(iii) AI = IA = A

(where A is square matrix and I is unit matrix of same order)

(iv) A x O = O x A = O

(where A is matrix of order m x n & O is matrix of order n x m)

(v) In General AB ≠ BA (Not Commutative)

(Note AB = BA only where B is equal to Adjoint of A)

(vi) IF AB = 0, doesn't imply that

A = 0 or B = 0 or both = 0

SELF EXAMINATION QUESTIONS

(1) Evaluate AB

where $A=\begin{bmatrix}0 & 1\\0 & 2\end{bmatrix}$ & $B=\begin{bmatrix}3 & 4\\0 & 0\end{bmatrix}$ $\left(\text{Ans.}\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\right)$

(2) Show that AI = IA = A

where $A=\begin{bmatrix}2 & 9\\6 & -5\end{bmatrix}$

(3) Evaluate AB & BA. Is AB = BA?

where $A=\begin{bmatrix}3 & 5\\4 & 2\end{bmatrix}$ $B=\begin{bmatrix}2 & 7\\8 & 9\end{bmatrix}$ (Ans.No.)

TRANSPOSE OF A MATRIX

Transpose of matrix A denoted by A' or A^t. Transpose of A can be obtained by inter changing rows and columns of a matrix A.

If A = $[a_{ij}]_{mxn}$ then A' = $[a_{ij}]_{nxm}$

Thus, If $A = \begin{bmatrix}3 & 2 & 4\\7 & 8 & 9\end{bmatrix}_{2x3}$

Then $A' = \begin{bmatrix}3 & 7\\2 & 8\\4 & 9\end{bmatrix}_{3x2}$

Properties of Transpose of a matrix

(i) (A)' = A

(ii) (A + B)' = A' + B'

(iii) (kA)' = kA' (Where k is a Scalar)

(iv) (AB)' = B'A'

SYMMETRIC MATRIX

Any matrix A is called to be symmetric matrix if A' = A

Example $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

SKEW SYMMETRIC MATRIX

Any matrix A is called to be skew symmetric if A' = -A

Example $A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & b & c \\ -b & 0 & c \\ -c & -e & 0 \end{bmatrix}$

Example 106 : Find A + A' where $A = \begin{bmatrix} 2 & 5 \\ 7 & 8 \end{bmatrix}$

& Prove A + A' is symmetric

Solution: $A + A' = \begin{bmatrix} 2 & 5 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 5 & 8 \end{bmatrix}$

$= \begin{bmatrix} 4 & 12 \\ 12 & 16 \end{bmatrix}$

Let $B = \begin{bmatrix} 4 & 12 \\ 12 & 16 \end{bmatrix}$ $(\therefore B = A + A')$

$B' = \begin{bmatrix} 4 & 12 \\ 12 & 16 \end{bmatrix} = B$

So, B =A+A', is symmetric matrix

Example 107 : Find A – A' where $A = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 7 & 6 \\ 8 & 2 & 9 \end{bmatrix}$

& Prove A – A' is skew symmetric.

Solution: $A - A' = \begin{matrix} 3 & 2 & 5 \\ 4 & 7 & 6 \\ 8 & 2 & 9 \end{matrix} - \begin{matrix} 3 & 4 & 8 \\ 2 & 7 & 2 \\ 5 & 6 & 9 \end{matrix}$

$= \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix} = B(\text{say})$

$$B' = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

$$= - \begin{matrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ 3 & -4 & 0 \end{matrix} \qquad B$$

So B = A – A' is skew symmetric matrix.

SELF EXAMINATION QUESTIONS

(1) Find A + A' where $A = \begin{bmatrix} 5 & 7 & 2 \\ -9 & 8 & 11 \\ 5 & 6 & -3 \end{bmatrix}$

& Prove (A + A') is symmetric.

(2) Find A – A' where $A = \begin{bmatrix} 11 & 7 & 9 \\ -8 & 6 & 5 \\ 6 & 2 & 13 \end{bmatrix}$

& Prove (A – A') is symmetric.

Example 108 : If $= \begin{bmatrix} 3x+y & 0 \\ 0 & x-y \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$

Then find x & y

Solution :

If two matrices are equal, then their corresponding elements are equal

$3x + y = 7$ (i)

$x - y = 1$ (ii)

Adding (i) & (ii)

$4x = 8$

$x = 2$

Put x in (i)

$3(2) + y = 7$

$y = 7 - 6 = 1$

$(x = 2 \;\&\; y = 1)$

Example 109 : If $3x + y = \begin{bmatrix} 2 & 7 \\ 8 & 9 \end{bmatrix}$

& $2x - y = \begin{bmatrix} 3 & 3 \\ 12 & 11 \end{bmatrix}$

Then find x & y

Solution:

$$3x + y = \begin{bmatrix} 2 & 7 \\ 8 & 9 \end{bmatrix} \qquad \text{(i)}$$

$$2x - y = \begin{bmatrix} 3 & 3 \\ 12 & 11 \end{bmatrix} \qquad \text{(ii)}$$

Adding (i) & (ii)

$$5x = \begin{bmatrix} 5 & 10 \\ 20 & 20 \end{bmatrix}$$

$$x = \frac{1}{5}\begin{bmatrix} 5 & 10 \\ 20 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix}$$

Put x in (i)

$$3\begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix} + y = \begin{bmatrix} 2 & 7 \\ 8 & 9 \end{bmatrix}$$

$$y = \begin{bmatrix} 2 & 7 \\ 8 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 12 & 12 \end{bmatrix}$$

$$y = \begin{bmatrix} -1 & 1 \\ -4 & -3 \end{bmatrix}$$

Example 110: Find $A^2 + 3A - 2I$

Where $A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$

Solution: $A^2 + 3A - 2I$

$$= \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} + 3\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 24 \\ 40 & 64 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ 15 & 21 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 33 \\ 55 & 83 \end{bmatrix}$$

SELF EXAMINATION QUESTIONS

(1) Find a, b, c & d

Where $\begin{bmatrix} 2a+b & c-d \\ 3a-b & 2c+3d \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ 13 & 9 \end{bmatrix}$

(Ans. a = 5 b = 2 c = 3 d = 1)

(2) Find X & Y

Where 3X + 2Y $= \begin{bmatrix} 3 & 7 \\ -1 & 2 \end{bmatrix}$

$X - Y = \begin{bmatrix} 1 & -6 \\ 8 & 9 \end{bmatrix}$

Ans. $X = \begin{matrix} 1 & -1 \\ 3 & 4 \end{matrix}$ $Y = \begin{matrix} 0 & 5 \\ -5 & -5 \end{matrix}$

(3) Find 'X'

Where AX = B

$A = \begin{bmatrix} 1 & 2 \\ 9 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 12 \\ 13 & 52 \end{bmatrix}$

$\left(\text{Ans. } x = \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix} \right)$

2.8.2. DETERMINANTS

The determinant of a square matrix is a number that associated with the square matrix. This number may be postive, negative or zero.

The determinant of the matrix A is denoted by det A or |A| or Δ

For 1 x 1 matrix A = [3]

|A| = 3

For matrix A = [–3]

|A| = – 3

For 2 x 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

For 3 x 3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$|A| = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$= a \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \begin{bmatrix} d & e \\ g & h \end{bmatrix}$

= a (ei – hf) –b (di – gf) + c(dh – ge)

Singular and Non Singular Matrix

If detA = 0 for square matrix, then it's a singular matrix.

If detA ≠ 0 for square matrix, then it's a non singular matrix.

Sub Matrix

If a finite number of rows or columns are deleted in a matrix then the resulting matrix is known as a sub matrix

Example 111 : $A = \begin{bmatrix} 3 & 2 & 4 \\ 5 & 6 & 7 \\ 8 & 2 & 1 \end{bmatrix}$

After the deletion of first row and first column of matrix A following sub matrix B obtain, where

$B = \begin{bmatrix} 6 & 7 \\ 2 & 1 \end{bmatrix}$

Example 112: If $A = \begin{matrix} 3 & 5 \\ 7 & 6 \end{matrix}$, find |A|

$|A| = 3 \times 6 - 7 \times 5$

$= 18 - 35 \Rightarrow -17$

Example 113 : If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, find |A|

Solution: |A|

$$=1\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$

$= 1(-3) - 2(-6) + 3(-3)$

$= -3 + 12 - 9$

$= 0$

Example 114 : Find the determinant of the given 3×3 matrix

$$\begin{matrix} 2 & -3 & -2 \\ -6 & 3 & 3 \\ -2 & -3 & -2 \end{matrix}$$

Solution:

$$\det\begin{vmatrix} 2 & -3 & -2 \\ -6 & 3 & 3 \\ -2 & -3 & -2 \end{vmatrix} = 2\det\begin{vmatrix} 3 & 3 \\ -3 & -2 \end{vmatrix} - (-3)\det\begin{vmatrix} -6 & 3 \\ -2 & -2 \end{vmatrix} + (-2)\det\begin{vmatrix} -6 & 3 \\ -2 & -3 \end{vmatrix}$$

$= 2\,[-6-(-9)] + 3\,[12-(-6)] - 2\,[18-(-6)]$

$= 2\,(3) + 3\,(18) - 2\,(24)$

$= \boxed{12}$

Example 115 : Find the determinant of the given 3×3 matrix

$$\begin{matrix} -4 & 5 & 2 \\ -3 & 4 & 2 \\ -1 & 2 & 5 \end{matrix}$$

Solution:

$$\det\begin{vmatrix} -4 & 5 & 2 \\ -3 & 4 & 2 \\ -1 & 2 & 5 \end{vmatrix} = -4\det\begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} - 4\det\begin{vmatrix} -3 & 2 \\ -1 & -5 \end{vmatrix} + 2\det\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix}$$

$= -4\,[20-4] - 5\,[-15-(-2)] + 2\,[-6-(-4)]$

$= -4\,(16) - 5\,(-13) + 2\,(-2)$

$= \boxed{-3}$

Example 116 : Find the determinant of the given 3×3 matrix

$$\begin{matrix} 1 & -3 & -6 \\ -1 & 5 & 5 \\ -1 & 6 & 5 \end{matrix}$$

Solution:

$$\det\begin{vmatrix} 1 & -3 & -6 \\ -1 & 5 & 5 \\ -1 & 6 & 5 \end{vmatrix} = 1\det\begin{vmatrix} 5 & 5 \\ 6 & 5 \end{vmatrix} - 3\,(-3)\det\begin{vmatrix} -1 & 5 \\ -1 & 5 \end{vmatrix} + (-6)\det\begin{vmatrix} -1 & 5 \\ -1 & 6 \end{vmatrix}$$

$= 1[25-30] + 3\,[-5\,(-5)] - 6\,[-6-(-5)]$

$= 1\,(-5) + 3\,(0) - 6\,(-1)$

$= \boxed{-1}$

Example 117 : Find the determinant of the given 3×3 matrix

$$\begin{matrix} 6 & 2 & -4 \\ 5 & 6 & -2 \\ 5 & 2 & -3 \end{matrix}$$

Solution:

$$\det \begin{matrix} 6 & 2 & -4 \\ 5 & 6 & -2 \\ 5 & 2 & -3 \end{matrix} = 6 \det \begin{matrix} 6 & -2 \\ 2 & -2 \end{matrix} - 2 \det \begin{matrix} 5 & -2 \\ 5 & -3 \end{matrix} + (-4) \det \begin{matrix} 5 & 6 \\ 5 & 2 \end{matrix}$$

$= 6 [- 18 - (- 4)] - 2 [- 15 (- 10)] - 4 [10 - 30]$

$= 6 (-14) - 2 (-5) - 4 (- 20)$

$= - 84 + 10 + 80$

$= \boxed{6}$

Example 118 : Find the determinant of the given 3×3 matrix

$$\begin{matrix} 5 & 3 & -4 \\ 2 & 0 & -2 \\ 2 & 5 & -1 \end{matrix}$$

Solution:

$$\det \begin{matrix} 5 & 3 & -4 \\ 2 & 0 & -2 \\ 2 & 5 & -1 \end{matrix} = 5 \det \begin{matrix} 0 & -2 \\ 5 & -1 \end{matrix} - 3 \det \begin{matrix} 2 & -2 \\ 5 & -1 \end{matrix} + (-4) \det \begin{matrix} 2 & 0 \\ 2 & 5 \end{matrix}$$

$= 5 [0 - (- 10)] - 3 [- 2 - (- 4)] - 4 [10 - 0]$

$= 5 (10) - 3 (2) - 4 (10)$

$= 50 - 6 - 40$

$= \boxed{4}$

SELF EXAMINATION QUESTIONS

(1) Evaluate $\begin{bmatrix} 8 & 7 \\ -3 & 2 \end{bmatrix}$ (Ans. 37)

(2) Evaluate $\begin{bmatrix} 4 & -2 & 1 \\ 7 & 3 & 3 \\ 2 & 0 & 1 \end{bmatrix}$ (Ans. 8)

Study Note - 3

CALCULUS

3.1 FUNCTION

FUNCTION :

If x and y be two real variables related to some rule, such that corresponding to every value of x within a defined domain we get a defined value of y, then y is said to be a function of x defined in its domain.

Here the variable x to which we may arbitrarily assign different values in the given domain is known as *inderpendent variable* (or argument) and y is called the *dependent variable* (or function).

Notations : Generally we shall represent functions of x by the symbols f(x), F(x), f(x), y(x) etc.

Example 1 : A man walks at an uniform rate of 5 km per hour. If s indicates the distances and t be the time in hours (from start), then we may write, s = 5t.

Here s and t are both variables,s is dependent if t is independent. Now s is a function of t and the domain (value) of t is $0 \le t \le \infty$.

Example 2 : $y = f(x) = \frac{x^2}{x}$.

For $x \ne 0$, y = x and for x = 0, y is not known (undefined). Here the domain is the set of real numbers *except zero*. (refer worked out problem 2 of limit & continuity)

Constant Function : y = f(x) = 7 for all real values of x. Here y has just one value 7 for all values of x.

Single-value, Multi-valued Function : From the definition of function we know that for y = f (x), there exists a single value of y for every value of x. This type of function is sometimes known as single-valued function.

Example 3 : y = f (x) = 2x + 3

For x = 1, y = 2.1 + 3 = 2 + 3 = 5.

= 2, y = 2.2 + 3 = 4 + 3 = 7

If again we get more than one value of y for a value of x, then y said to be a *multiple-valued* (or multi-valued) function of x.

Example 4 : $y^2 = x$. Here for every $x > 0$, we find two values of y as $y = \pm\sqrt{x}$.

Explicit and Implicit Function : A function is said to be *explicit* when it is expressed directly in terms of the independent variable ; otherwise it is implicit.

Example 5 : $y = x^2 - x + 1$ is an explicit function :

$2x^2 + 3xy + y^2 = 0$ an implicit function.

Parametic Representation of a Function : If the dependent variable x be expressed in terms of a third variable, say t, i.e., $y = f(t)$, $x = F(t)$, then these two relations together give the parametic representation of the function between y & x.

Example 6 : $y = t^2 + 1$, $x = 2t$.

Odd and Even Functions : A function f (x) is an odd function of x if $f(-x) = -f(x)$ and is an even function of x if $f(-x) = = f(x)$.

Example 7 : $f(x) = x$. Now $f(-x) = -f(x)$, so $f(x) = x$ is an odd function of x.

$f(x) = x^2$, $f(-x) = (-x)^2 = x^2 = f(x)$, so $f(x) = x^2$ is an even function of x.

Inverse Function : It forms a function $y = f(x)$, we can obtain another function $x = F(y)$, then each functions known as the inverse of the other.

Example 8 : $y = 4x - 3$ and $x = \frac{1}{4}(y+3)$ are inverse to each other.

Functions of one independent variable :

Polynomial Function : A function of the form

$$F(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n,$$

Where n is a positive integer and $a_0, a_1 \ldots . a_n$ are constants is known as a polynomial function in x.

For $n = 0$, $f(x) = a_0$, a constant function

$= 1$, $f(x) = a_0 + a_1x$, a linear function in x

$= 2$, $f(x) = a_0 + a_1x + a_2x^2$, a quadratic function in x.

$= 3$, $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, a cubic function in x.

Rational function : A function that is expressed as the ratio of two polynomials

i.e., $f(x) = \frac{a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n}{b_0 + b_1x + b_2x^2 + \ldots\ldots + b_nx^n}$.

i.e., in the form of $\frac{P(x)}{Q(x)}$

is called a rational function of x, such function exists for denominator $\neq 0$.

Example 9 : $f(x) = \frac{x+2}{x^2 - 4x + 3}$ exists for all values of x, if $x^2 - 4x + 3 \neq 0$. Now for $x^2 - 4x + 3 = 0$ or

$(x - 1)(x - 3) = 0$ or, $x = 1, 3$ Denominator becomes zero and hence the given function does not exist.

Irrational function : On the contrary if a function f(x) can not be represented in this form, it is called an irrational function.

Example 10 : Functions of the form $\sqrt{x}$ (where x is not a square number)

Algebraic function : A function in the form of a **polynomial** with **finite number of terms** is known as algebraic function.

Example 11 : $x^2 + 2x - 3, \sqrt{x^2 + 1}$ etc.

Domain and Range of a Function :

The set of values of independent variables x is called the '*Domain*' of the function and the set of corresponding values of f(x) i.e. the dependent variable y is called the '*Range*' of the function.

Example 12 : For the squared function of $y = x^2$, we get the ordered pairs (1, 1) (2, 4) (3, 9), ..., The set of independent variables {... –2, –1, 0, 1, 2, 3, ...} is the domain, where as set of dependent variable {0, 1, 4, 9, ...} represents the range.

Example 13 : For the following functions find the domain and range.

(i) $f(x) = \frac{x^2 - 4}{x - 2}, x \neq 2$ (ii) $f(x) = \frac{3 - x}{x - 3}, \ x \neq 3$ (iii) $f(x) = \frac{2x + 1}{(x - 1)(x + 1)}$

Solution:

(i) $f(0) = \frac{-4}{-2} = 2, f(1) = \frac{-3}{-1} = 3, f(3) = \frac{5}{1} = 5, f(4) = 6, f(-1) = 1$

$\therefore$ domain = { –1, 0, 1, 3, 4, ...}, range = {1, 2, 3, 5, 6, ...}

= R – {2}, = R – {4}, R = real number,

(ii) $f(0) = \frac{3}{-3} = -1, f(1) = -1, f(2) = -1, f(-1) = -1$

domain = { – 1, 0, 1,2, 4, ...}, range = { –1, –1, –1, ...} = {–1}

= R – {3}, where R is a real number

(iii) The value of f(x) exists for $x \neq 1, x \neq -1$.

$\therefore$ domain = R – {– 1, 1}, i.e, all real numbers excluding 1 & (–1)

range = R.

Example 14 : Find the domain of definition of the function $\frac{4x - 5}{\sqrt{x^2 - 7x + 12}}$

Denominator $= \sqrt{x^2 - 7x + 12} = \sqrt{(x - 3)(x - 4)}$. Given function cannot be defined for $3 \leq x \leq 4$. So domain is for all real values of x except 3 and 4.

Absolute Value : A real number "a" may be either a = 0, or a > 0 or, a < 0. The absolute value (for modulus) of a, denoted by | a | is defined as | a | = a, for a > 0

= – a, for a < 0

Thus | – 4| = – (– 4) = 4, and | 4 | = 4.

Complex No :

A number of the form (a+ib) or (a–ib); where a & b are real numbers is called a complex number (where $i = \sqrt{-1}$)

The complex number has two parts; a real part & an imaginary part. 'a' is the real part & 'ib' is the imaginary part.

Example 15 :

If $y = x^2 + 4$ is a function under consideration then solving for y = 0, we get

$0 = x^2 + 4$

or, $x^2 = -4$

or, $x = \pm\sqrt{-4}$

or, $x = \pm 2\sqrt{-1} = \pm 2i$

The number ± 2i is a complex number whose real part is 0 & imaginary part is ±2

Example 16 : Given $f(x) = 2x^2 - 3x + 1$; find f(2), f(0), f (– 3)

$f(2) = 2.2^2 - 3.2 + 1 = 2.4 - 6 + 1 = 8 - 6 + 1 = 3$

$f(0) = 2.0^2 - 3.0 + 1 = 2.0 - 0 + 1 = 0 - 0 + 1 = 1$

$f(-3) = 2(-3)^2 - 3.(-3) + 1 = 2.9 + 9 + 1 = 18 + 9 + 1 = 28.$

Example 17 : If y = 4x – 1, find the value of y for x = 2. Can y be regarded as a function of x? Also find the domain.

For x = 2, y = 4.2 – 1 = 8 – 1 = 7. Again for x = 0, y = – 1 and for x = – 1, y = – 5. So for every value of x in $-\infty < x < \infty$, we find different values of y, So y is a function of x and its domain is -

$-\infty < x < +\infty$.

Example 18 : If f (x) = x + | x |, find f (3) and f (– 3) and show also they are not equal.

f(3) = 3 + | 3 | = 3 + 3 = 6 ; f (– 3) = – 3 + | – 3| = – 3 + 3 = 0.

As 6 ≠ 0, so f(3) ≠ f (– 3).

Note : If f(x) = f(– x) [i.e., f (3) = f (– 3)] then f(x) will be an even function of x.

Example 19 : Show that $\sqrt{x^2 - 5x + 4}$ is not defined for 1 < x < 4

$$\sqrt{x^2 - 5x + 4} = \sqrt{(x-1)(x-4)}.$$

Now for any value x > 1, but < 4 the expression becomes imaginary. So the expression is undefined for 1 < x < 4.

Example 20 : Find the domain of $f(x) = \dfrac{x}{x^2 - 9}$

Here f(x) has a unique value except for x = 3, – 3.

For $f(3) = \dfrac{3}{9-9} = \dfrac{3}{0}$ (undefined) and $f(-3) = \dfrac{-3}{9-9} = \dfrac{-3}{0}$ (undefined)

∴ domain of the function f(x) is $-\infty < x < -3$; $-3 < x < 3$ and $3 < x < \infty$.

Example 21 : Given the function

$$f(x) = 5^{-2x} - 1, \quad -1 \le x < 0$$

$$= \frac{x^2 - 2}{x - 2}, \quad 0 \le x < 1$$

$= \frac{2x}{x^2 - 1}, \quad 1 \le x < 3$

Find f (– 1), f(0), f (1/2), f (2).

$f(-1) = 5^{-2(-1)} - 1$ (since – 1 lies in – 1 ≤ x < 0)

$= 5^2 - 1 = 25 - 1 = 24.$

Points 0, $\frac{1}{2}$ lie in the second interval ; so,

$$f(0) = \frac{0-2}{0-2} = 1, \quad f\left(\frac{1}{2}\right) = \frac{\frac{1}{4} - 2}{\frac{1}{2} - 2} = \frac{7}{6}.$$

Now 2 lies in the third interval $f(2) = \frac{2.2}{2^2 - 1} = \frac{4}{4-1} = \frac{4}{3}.$

Example 22 : If $f(x) = e^{ax+b}$. Prove that $e^b f(x + y) = f(x). f(y)$

$e^b f(x + y) = e^b . e^{a(x+y)+b} = e^{b+ax+ay+b}$

$= e^{ax+b} . e^{ay+b} = f(x). f(y)$. Hence proved.

Example 23 : If f(x) = x – a, q(x) = x + a then show that

$\{f(x)\}^2 - \{q(x)\}^2 = -2a \{f(x) + q(x)\}$

L.H.S. $= (x - a)^2 - (x + a)^2 = x^2 + a^2 - 2ax - (x^2 + a^2 + 2ax) = -4ax$

R.H.S. $= -2a \{x - a + x + a\} = -2a. 2x = -4ax$.

SELF EXAMINATION

1. If f(x) = (x – 1) (x – 2 (x – 3) find the values of

(i) f(1) (ii) f(2) (iii) f(3) (vi) f(0). [Ans. 0, 0, 0, – 6]

2. If f(x) = | x | + x, find whether f (3) and f(– 3) are equal? [Ans. No]

3. Given f(x) = x, and $F(x) = \frac{x^2}{x}$ is F(x) = f(x) always? [Ans. equal, except x = 0]

4. (i) If $f(x) = x^2 + 2x^4$ verify f(x) = f (– x).

(ii) If $f(x) = x + 2x^3$ verify f(x) = – f (– x)

5. If y = 5 for every value of x, can y be regarded as a function of x? [Ans. Yes]

6. f(x) = x + | x |, are f(4) and f(– 4) equal? [Ans. No]

7. If $f(x) = b\frac{x-a}{b-a} + a\frac{x-b}{a-b}$, then f(a) + f(b) = f(a + b).

8. If $f(x) = x^2 - x$, then prove that f(h+1) = f(– h)

9. If $\phi(x) = a\frac{x-b}{a-b} + b\frac{x-a}{b-a}, (a \ne b)$ obtain φ (a), φ (b) and φ (a + b) and then verify φ (a) + φ (b) = φ (a + b).

10. If $y = f(x) = \frac{ax+b}{cx-a}$ prove that φ(y) = x.

11. Show that $\frac{x^2-6x+8}{x^2-8x+12}$ is undefined for x = 2 and also find f(6). [Ans. undefined]

12. For what values of x are the following functions not defined?

(i) $\frac{1}{x-4}$, (ii) $\sqrt{x}$, (iii) $\sqrt{x-1}$, (iv) $\sqrt{(x-1)(x-2)}$, (v) $\frac{x^2-1}{x-1}$

[Ans. (i) 4, (ii) any negative integer, (iii) x < 1, (iv) 1 < x < 2, (v) x = 1]

13. Let f(x) $= 2^{-x}$ $\quad -1 \le x < 0$

$= 4 \quad 0 \le x < 1$

$= 2x - 1, \quad 1. \le x \le 3$

Calculate f(– 1), f(0), f(1), f(3). [Ans. 2, 4, 1, 5]

14. A 3-wheeler charges Re. 1 for 1 km or less from start, and at a rate of 50 p. per km or any fraction thereof, for additional distance. Express analytically the fare F (in ₹) as a function of the distance d (in km). [Ans. F = 1, 0 < d < 1, $F = 1 + \frac{1}{2}(d-1)$, d > 1]

15. If $g(x) = 2^x$, then show that g(a), g(b) = g(a + b).

16. If $f(x) = e^{px+q}$, (p, q are constants) then show that $f(a)\,f(b)\,f(c) = f(a+b+c)\,e^{2q}$.

17. If $f(x) = \frac{ax+b}{bx+a}$, prove that $f(x).\ f\ \frac{1}{x} = 1$.

18. If $f(x) = \frac{2x+1}{2x^2+1}$ and f (x) = 2f (2x) then find f(2, 5). [Ans. 22/51]

[**Hints :** $f(x) = 2.\frac{2(2x)+1}{2.(2x)^2+1} = \frac{8x+2}{8x^2+1}$ & etc.]

3.2 LIMIT

LIMIT :

Introduction :

Calculus is based, in general, on the idea of a limit. At present this idea including its related concepts, continuity to mention, will be discussed.

Some definitions:

(i) **Meaning of "x tends to a".** When the difference |x–a| (i.e., numerical difference) can be made less than any positive quantity, however small, we say x tends to a and is written as $x \to a$.

(ii) **Meaning of "x tends to zero".** When the value of x goes on decreasing numerically and can be made numerically less than any positive quantity, however small, we say x tends to zero and is written as $x \to 0$.

(iii) **Meaning of "x tends to infinity".** When the value of x goes increasing and can be made greater than any positive quantity, however, large, we say x tends to infinity written as $x \to \propto$.

Neighbourhood or Proximity of a point

Let c be any real number, then any open interval around c is called the neighbourhood of c, e.g.,

$$]c-\in, c+\in[\,, \in > 0$$

Is the neighbourhood of c.

Any neighbourhood from which the point c is excluded is called deleted neighbourhood of c.

Geometrically it means set of those points which are within an $\in$ distance from c on either side except for the point c.

(iv) **Limit of a function f (x).** A number l is said to be the limit of f(x) as x→a written as $\underset{x\to a}{Lt}\ f(x) = l$ if

a. The function is defined and single valued in the deleted neighbourhood of a.

b. For every positive number $\in$, however small, there exists a positive number δ (usually depending on $\in$), such

$$|f(x) - 1| < \in$$

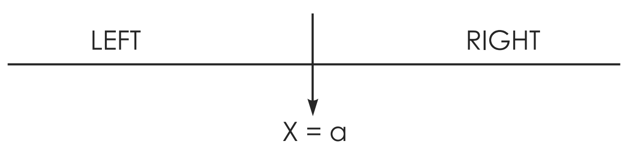
Whenever $$0 < |x - a| < \delta$$

From the above definition it is interesting to note that $\underset{x\to a}{Lt}\ f(x)$ may exist, even if the function f(x) is not defined at x = a. Sometimes both the things may happen, i.e.,

(i) The function is defined at x = a, and

(ii) $\underset{x\to a}{Lt}\ f(x)$ also exists.

Right hand and left hand limits



The variable point x can approach a either from the left or from the right. These respective approaches are indicated by writing

$x \to a-$

and $x \to a+$

If $\underset{x\to a-}{Lt}\ f(x) = l_1$ (left hand limit)

and $\underset{x\to a+}{Lt}\ f(x) = l_2$ (right hand limit)

we say $\underset{x\to a-}{Lt}\ f(x) = l$ if and only if

$$l_1 = l_2 = l$$

Methods of finding limit of a function f(x) as x tends to a finite quantity say 'a'

There are three methods for finding limit of a function f(x) as x tends to a finite quantity say 'a' :

(i) Method of factors

(ii) Method of substitution

(iii) Method of rationalization.

In method I, if f(x) is of the form $\frac{g(x)}{h(x)}$ factorise g(x) and h(x), cancel the common factors and then put the value of x.

Example 24: *Find the value of* $\underset{x\to 1}{Lt}\frac{x^3-1}{x^2-1}$.

Solution:

Now, $\frac{x^3-1}{x^2-1}=\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$

$$=\frac{x^2+x+1}{x+1}$$

$\therefore \quad \underset{x\to 1}{Lt}\frac{x^3-1}{x^2-1}=\underset{x\to 1}{Lt}\frac{x^2+x+1}{x+1}$

$$=\frac{1+1+1}{1+1}=\frac{3}{2}.$$

Method II

The following steps are involve:

(i) Put x = a + h where h is very small but ≠ 0, i.e. x → a, h→0.

(ii) Simplify numerator and denominator and cancel common powers of h.

(iii) Put h = 0.

The result is the required limit.

Example 25: Evaluate $\underset{x\to a}{Lt}\frac{x^n-a^n}{x-a}$.

Solution:

Put x = a + h where h is very small, then

$$\frac{x^n-a^n}{x-a}=\frac{(a+h)^n-a^n}{h}$$

$$=\frac{a^n\left[\left(1+\frac{h}{a}\right)^n-1\right]}{h}$$

$$=\frac{a^n\left[1+\frac{nh}{a}+n(n-1)\frac{h^2}{a^2}+\cdots-1\right]}{h}$$

$$=\frac{a^n\left[n\frac{h}{a}+n(n-1)\frac{h^2}{a^2}+\cdots\right]}{h}$$

$$= a^n \quad \frac{n}{a} + n(n-1)\frac{h}{a^2} + \cdots$$

$$\therefore \qquad \underset{x \to a}{\text{Lt}} \frac{x^n - a^n}{x - a} = \underset{h \to 0}{\text{Lt}}\ a^n \quad \frac{n}{a} + n(n-1)\frac{h}{a^2} + \cdots$$

$$= a^n \cdot \frac{n}{a} = na^{n-1}$$

Method III. (Rationalisation)

This method is useful where radical sings are involved either in the numerator or denominator. The numerator or denominator (as required) is rationalized and limit taken. The following example will make the method clear.

Example 26: Evaluate $\underset{x \to 0}{\text{Lt}} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

Solution:

Rationalising the numerator, we get

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \frac{\sqrt{1+x} - \sqrt{1+x}}{x} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \frac{2x}{x\ \sqrt{1+x} + \sqrt{1-x}}$$

$$= \frac{2}{\sqrt{1+x} + \sqrt{1-x}}$$

$$\therefore \qquad \underset{x \to 0}{\text{Lt}} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \underset{x \to 0}{\text{Lt}} \frac{2}{\sqrt{1+x} - \sqrt{1-x}} = 1$$

Infinite Limits

For finding the limit of $f(x) = \frac{g(x)}{h(x)}$ as $a \to \infty$, we divided the numerator and denominator by highest power of x occurring in f(x) (numerator or denominator whichever power is high) and then use $\frac{1}{x}, \frac{1}{x^2}$, etc.$\to 0$ as $x \to \infty$.

Example 27: Evaluate

$$\underset{x \to \infty}{\text{Lt}} \frac{(x+1)(2x+3)}{(x+2)(3x+4)}$$

Solution:

Now $\frac{(x+1)(2x+3)}{(x+2)(3x+4)} = \frac{2x^2+5x+3}{3x^2+10x+8}$

$$= \frac{2+\frac{5}{x}+\frac{3}{x^2}}{3+\frac{10}{x}+\frac{8}{x^2}} \quad \text{[Dividing by } x^2\text{]}$$

$\therefore \quad \underset{x\to\infty}{Lt} \frac{(x+1)(2x+3)}{(x+2)(3x+4)} = \frac{2}{3}$

Or $\quad \underset{x\to\infty}{Lt} \frac{(x+1)(2x+3)}{(x+2)(3x+4)}$

$$= \underset{x\to\infty}{Lt} \frac{\left(1+\frac{1}{x}\right)\left(2+\frac{3}{x}\right)}{\left(1+\frac{2}{x}\right)\left(3+\frac{4}{x}\right)} = \frac{1\cdot 2}{1\cdot 3} = \frac{2}{3}$$

SOLVED EXAMPLES

Example 28 : $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$

For if x = 2 + h, whether h be positive or negative,

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{(x-2)} = \frac{h(4+h)}{h} = 4+h$$

and by taking h numerically small, the difference of $\frac{x^2-4}{x-2}$ and 4 can be made as small as we like. It may be noted here that however small h may be, as h ≠ 0, one can cancel the factor (x – 2) i.e., h between numerator and denominator here. Hence $\lim_{x\to 2} \frac{x^2-4}{x-4} = 4.$ But for x = 2, the function $\frac{x^2-4}{x-2}$ is undefined as we cannot cancel the factor x – 2, which is equal to zero.

Now writing $f(x) = \frac{x^2-4}{x-2}, \lim_{x\to 2} f(x) = 4$ whereas f(2) does not exist.

Example 29 : For f (x) = | x |, find, $\lim_{x\to 0} f(x)$.

Solution:

f(x) = | x | means $\begin{matrix} x, \text{ for } x \geq 0 \\ -x, \text{ for } x < 0 \end{matrix}$

Now $\quad \lim_{x\to 0+} f(x) = \lim_{x\to 0+} |x| = \lim_{x\to 0+} x = 0$

And $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} |x| = \lim_{x\to 0-}(-x) = 0.$

$\therefore \quad \lim_{x\to 0} f(x) = 0.$

Example 30 : Find analytically $\lim_{x\to 3}\sqrt{x-3}$, if it exists.

$\lim_{x\to 3+}\sqrt{x-3} = 0,$ but $\lim_{x\to 3-}\sqrt{x-3}$ does not exist. As when $x \to 3-0$ is definitely less than 3, whereas $x-3$ is negative and square root of negative is never real but an imaginary.

$\therefore$ the limit does not exist.

Alternatively : Put $x = 3 + h$, as $x \to 3$, $h \to 0$

$$\lim_{x\to 3}\sqrt{x-3} = \lim_{h\to 0}\sqrt{3+h-3} = \lim_{h\to 0}\sqrt{h}$$

$\lim_{h\to 0+}\sqrt{h} = 0$, but $= \lim_{h\to 0-}\sqrt{h}$ does not exist (as the h near to but less than zero corresponds no real value of $\sqrt{h}$).

$\therefore \quad \lim_{x\to 3}\sqrt{x-3}$ does not exist.

Note : At $x = 3$, $f(x) = \sqrt{x-3} = \sqrt{3-3} = 0 \ \therefore$ f(3) exists.

Example 31 : Do the following limits exists? If so find the values

Solution:

(i) $\lim_{x\to -2}\frac{1}{x+2}$ (ii) $\lim_{x\to 0}\frac{1}{x}$ (iii) $\lim_{x\to 1}\ (x^2-1)+\frac{(x-1^2)}{x-1}$

(i) $\lim_{x\to -2}\frac{1}{x+2} = \lim_{h\to 0}\frac{1}{-2+h+2} = \lim_{h\to 0}\frac{1}{h}$

Now $\lim_{x\to 0+}\frac{1}{h} = +ve\,;\ \lim_{h\to 0-}\frac{1}{h} = -ve.$ As the two values are not same, so the limit does not exist.

(ii) $\lim_{x\to 0+}\frac{1}{x} = +\infty,\ \lim_{h\to 0-}\frac{1}{x} = -\infty.$ The limit doesn't exist as the two values are unequal.

(iii) Expr. $\lim_{x\to 1}\ (x^2-1)+\frac{(x-1)(x-1)}{(x-1)} = \lim_{x\to 1}\{(x^2-1)+(x-1)\},=$ as $(x-1) = 0$;

$= \lim_{x\to 1}(x^2+x-2) = (1+1-2) = 0.$ On putting the limiting value of x, the value of the function exists and its value is 0.

Distinction between $\lim_{x\to a} f(x)$ and f(a). By $\lim_{x\to a} f(x)$ we mean the value of f(x) when x has any arbitary value near a but not a. The quantity f (a) is the value of f(x), when x is exactly equal to a.

Note : The following cases may arise :

(i) f(a) does not exist, but $\lim_{x \to a} f(x)$ exists.

(ii) f(a) exists, but $\lim_{x \to a} f(x)$ does not exist.

(iii) f(a) and $\lim_{x \to a} f(x)$ both exist, but unequal.

Let f(x) = 0, for x ¹ 0
= 1, for x = 0

$\lim_{x \to 0+} f(x) = 0 = \lim_{x \to 0-} f(x)$ and f(0) = 1, unequal values.

(iv) f (a) and $\lim_{x \to a} f(x)$ both exist and equal.

(v) neither f(a) or, $\lim_{x \to a} f(x)$ exists.

Fundamental Theorem on Limits :

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} \phi(x) = m,$ where *l* and m are finite quantities then

1. $\lim_{x \to a} f(x) \pm \phi(x) = l \pm m$

2. $\lim_{x \to a} f(x).\, \phi(x) = lm.$

3. $\lim_{x \to a} \frac{f(x)}{\phi(x)} = \frac{l}{m}, m \neq 0$

4. If $\lim_{x \to a} \phi(x) = b$ and $\lim_{x \to b} f(u) = f(b)$ then

$$\lim_{x \to b} f\,\{\phi(x)\} = f\left\{\lim_{x \to a} \phi(x)\right\} = f(b).$$

Example 32 : Evaluate, $\lim_{x \to 1} \frac{x^2 + 3x - 1}{2x + 4}.$

Solution:

As the limit of the denominator ≠ 0, we get

$$\text{Expression} = \frac{\lim_{x \to 1} \left(x^2 + 3x - 1\right)}{\lim_{x \to 1} (2x + 4)} = \frac{\lim_{x \to 1} x^2 + \lim_{x \to 1} 3x - \lim_{x \to 1} 1}{\lim_{x \to 1} 2x + \lim_{x \to 1} 4} \quad \text{(by therom 1)}$$

$$= \frac{1 + 3.1 - 1}{2.1 + 4} = \frac{3}{6} = \frac{1}{2}$$

We have not applied the definition to save labour. If we substitute x = 1, we get the value of the function $= \frac{1}{2}$ (equal to the limit the value as $x \to 1$). Practically this may not happen always, as shown below.

Example 33 : Evaluate $\lim_{x\to 1}\frac{x^2-3x+2}{x^2-4x+3}$.

Reqd. limit = $\lim_{x\to 1}\frac{(x-1)(x-2)}{(x-1)(x-3)}=\lim_{x\to 1}\frac{x-2}{x-3}=\frac{1-2}{1-3}=\frac{1}{2}$.

If we put x = 1, the function becomes $\frac{0}{0}$ which is undefined. Further limit of the denominator is zero, we cannot apply theorem 2, hence cancelling the common factor (x – 1) which is ≠ 0 as x →1, we may obtain the above result.

Problem when variable tends of infinity :

Example 34 : Find the limit of $\lim_{x\to\infty}\left(2x^2-5x+2\right)$

Solution:

Now $2x^2 - 5x + 2 = x^2$

$$\therefore\ \lim_{x\to\infty}\left(2x^2-5x+2\right)=\lim_{x\to\infty}x^2\ \ 2-\frac{5}{x}+\frac{2}{x^2}\quad=\lim_{x\to\infty}x^2\times\lim_{x\to\infty}\ \ 2-\frac{5}{x}+\frac{2}{x^2}$$

Now $\lim_{x\to\infty}x^2=\infty$; $\lim_{x\to\infty}2=2$; $\lim_{x\to\infty}\frac{5}{x}=0$; $\lim_{x\to\infty}\frac{2}{x^2}=0$

$$\therefore\ \lim_{x\to\infty}\left(2x^2-5x+2\right)=\infty\times 2=\infty.$$

Example 35 : Find $\lim_{x\to\infty}\frac{4x^5+2x^3-5}{7x^8+x^4+2}$

Solution:

Here the highest power of x in the denominator is 8. We now divide both the numberator and the denominator by x^8 to avoid the undefined form $\frac{\infty}{\infty}$. So we get

$$\lim_{x\to\infty}\frac{\frac{4}{x^3}+\frac{2}{x^5}-\frac{5}{x^8}}{7+\frac{1}{x^4}+\frac{2}{x^8}}=\lim_{u\to 0}\frac{4u^3+2u^5-5u^8}{7+u^4+2u^8}\quad \left(\text{put } \frac{1}{x}=u,\ \text{so as } x\to\infty\ ,\ u\to 0\right)$$

$$=\frac{\lim_{u\to 0}\left(4u^3+2u^5-5u^8\right)}{\lim_{u\to 0}\left(7+u^4+2u^8\right)}\qquad (\text{as denominator} \neq 0)$$

$$=\frac{4\lim_{u\to 0}u^3+2\lim_{u\to 0}u^5-5\lim_{u\to 0}u^8}{7+\lim_{u\to 0}u^4+2\lim_{u\to 0}u^8}=\frac{0+0-0}{7+0+0}=\frac{0}{7}=0$$

Example 36 : Find $\lim_{x\to\infty} \frac{5-2x^2}{3x+5x^2}$.

Solution:

$$\text{Expression} = \lim_{x\to\infty} \frac{\frac{5}{x^2}-2}{\frac{3}{x}+5} = \lim_{u\to 0} \frac{5u^2-2}{3u+5} \qquad \frac{1}{x}=u, \text{ as } x\to\infty, u\to 0$$

$$= \frac{5\lim_{u\to 0} u^2-2}{3\lim_{u\to 0} u+5} = \frac{0-2}{0+5} = -\frac{2}{5}.$$

Problems regarding rationalisation :

Example 37 : Find the value of : $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}, \quad (h\neq 0) \ = \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Problem regarding both sides limits :

Example 38 : A function is defined as

f(x)	$= x^2$,	for	$x > 1$
	$= 4.1$,	for	$x = 1$
	$= 2x$,	for	$x < 1$.

Does $\lim_{x\to 1} f(x)$ exist?

Solution:

Let us find $\lim_{x\to 1+} f(x)$ and $\lim_{x\to 1-} f(x)$.

Now $\lim_{x\to 1+} f(x) = \lim_{x\to 1+} x^2 = 1^2 = 1,$ as $f(x) = x^2$ for $x > 1$

Again $\lim_{x\to 1-} f(x) = \lim_{x\to 1-} 2x = 2.1 = 2,$ as $f(x) = 2x$ for $x < 1$

Since the two limiting values are not equal, so $\lim_{x\to 1} f(x)$ doesn't exist.

Example 39 : Evaluate (i) $\lim_{x\to 0}\frac{4x+|x|}{3x+|x|}$ (ii) $\lim_{x\to 2}\frac{|x-2|}{x-2}$

Solution:

(i) $\lim_{x\to 0+}\frac{4x+x}{3x+x}=\lim_{x\to 0}\frac{5x}{4x}=\frac{5}{4}$, as $x\neq 0$ and also for $x>0$, $|x|=x$

$\lim_{x\to 0-}\frac{4x-x}{3x-x}=\lim_{x\to 0-}\frac{3x}{2x}=\frac{3}{2}$ as $x\neq 0$, and for $x<0$, $|x|=-x$

$\therefore$ the given limit doesn't exist, as the above two limiting values are unequal.

(ii) $\lim_{x\to 2+}\frac{|x-2|}{x-2}=\lim_{x\to 2+}\frac{x-2}{x-2}=1.$

for $|x-2|>0$ i.e., for $x>2$, $|x-2|=x-2$

$\lim_{x\to 2-}\frac{|x-2|}{x-2}=\lim_{x\to 2-}\frac{-(x-2)}{(x-2)}=-1$

for $|x-2|<0$ i.e., for $x<2$, $|x-2|=-(x-2)$

As the two limiting values are unequal, so the given expression does not exist.

Some Useful Limits :

(A) $\lim_{x\to 2}(1+x)^{1/x}=e$

(B) $\lim_{x\to 0}\frac{1}{x}\log_e(1+x)=1$

(C) $\lim_{x\to 0}\frac{e^x-1}{x}=1$

(D) $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$

(E) $\lim_{x\to a}\frac{x^n-a^n}{x-a}=an^n-1$

(F) $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=n.$

Example 40 : $\lim_{x\to 0}\frac{e^{3x}-1}{x}=3.$ Put $3x=u$, as $x\to 0$ $u\to 0$.

$$\lim_{x\to 0}\frac{e^{3x}-1}{x}=\lim_{u\to 0}\frac{e^u-1}{u/3}=3\times\lim_{u\to 0}\frac{e^u-1}{u}.=3\times 1(\text{by C})=3.$$

Example 41 : $\lim_{x\to 0}\frac{\log(1+6x)}{x}=6.$

Solution:

Put $6x=u$, as $x\to u$, $u\to 0$

Expression $=\lim_{u\to 0}\frac{\log(1+u)}{u/6}=6\times\lim_{u\to 0}\frac{\log(1+u)}{u}=6\times 1$ (by B) $=6.$

Example 42 : $\lim_{x\to0}\frac{a^x-b^x}{x}$

Solution:

Expression $=\lim_{x\to0}\frac{a^x-1-b^x+1}{x}=\lim_{x\to0}\frac{a^x-1}{x}-\lim_{x\to0}\frac{b^x-1}{x}$

$=\log a-\log b$ [by (D) above] $=\log\frac{a}{b}$.

Example 43 : $\lim_{x\to0}\frac{e^{ax}-e^{bx}}{x}$

Solution:

Expression $=\lim_{x\to0}\frac{(e^{ax}-1)-(e^{bx}-1)}{x}=\lim_{x\to0}\frac{e^{ax}-1}{x}-\lim_{x\to0}\frac{e^{bx}-1}{x}$

$=\lim_{u\to0}\frac{e^u-1}{u/a}-\lim_{v\to0}\frac{e^v-1}{v/b}$

$=a\lim_{u\to0}\frac{e^u-1}{u}-b\lim_{v\to0}\frac{e^v-1}{v}$

$=a.\,1-b.1=a-b.$

SELF EXAMINATION QUESTION

Show that :

1. (i) $\lim_{x\to1}\frac{x^2-1}{x-1}=2$ (ii) $\lim_{x\to2}\frac{x^2-4}{x-2}=4$ (iii) $\lim_{x\to-2}\frac{x^2-4}{x+2}=-4$

2. (i) $\lim_{x\to2}\frac{3x}{x+1}$ [Ans. 2] (ii) $\lim_{x\to3}(x+1)(x+2)$ [Ans. 20]

(iii) $\lim_{x\to0}(x^2+3x-7)$ [Ans.–7] (iv) $\lim_{x\to0}\frac{h^2+4h}{2h}$ [Ans. 2]

(v) $\lim_{x\to2}\frac{3x^2-4x+7}{3x-5}$ [Ans. 11] (vi) $\lim_{x\to2}\frac{(2x-1)^2}{x+1}$ [Ans. 3]

3. (i) $\lim_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$ [Ans.1] (ii) $\lim_{x\to0}\frac{\sqrt{1+x^2}-1}{x}$ [Ans. 0]

(iii) $\lim_{x\to0}\frac{1-\sqrt{1-x^2}}{x^2}$ [Ans. $\frac{1}{2}$] (iv) $\lim_{x\to0}\frac{\sqrt{1+2x}-\sqrt{1-3x}}{x}$ [Ans. $\frac{5}{2}$]

Do the following limits exists? If so find the values.

4. (i) $\lim_{x\to3}f(x)$, when f (x) = 2x + 3, x > 3

= 3x + 1, x £ 3. [Ans. doesn't exist]

(ii) $\lim_{x \to 0} f(x)$ when f(x) = 1, $x\ ^{3}\ 0$.

= – 1, x < 0

[Ans. doesn't exist, in graph (0, – 1) will be excluded]

5. A function f(x) is defined as follow :

$f(x) = x^2$, for x < 1.

= 2. 5, for x = 1

$= x^2 + 2$, for x > 1.

Does $\lim_{x \to 1} f(x)$ exist? [Ans. no]

6. Given f(x) $= x^2$, for x > 1

= 2. 1, for x = 1

= x, for x < 1.

Find $\lim_{x \to 1} f(x)$. [Ans. 1]

Verify the following limits :

7. (i) $\lim_{x \to \infty} \frac{4x^3 + 5x - 1}{6x^3 + 7x^2 + 4} = \frac{2}{3}$. (ii) $\lim_{x \to \infty} \frac{6 - 5x^2}{4x + 15x^2} = -\frac{1}{3}$.

(iii) $\lim_{x \to \infty} \frac{4x^5 - x + 7}{2x^7 + 3x + 2} = 0$. (iv) $\lim_{x \to \infty} \frac{2x^9 - 5x^2 + 2}{9x^5 + 2x + 12} = \infty$

8. (i) $\lim_{x \to 0} \frac{e^{x^2 - 1}}{x}$. [Ans. 0] (ii) $\lim_{x \to 1} \frac{\log x}{x - 1}$ [Ans. 1]

3.3 CONTINUITY

Introduction :

A function is said to be continuous if its graph is a continuous curve without any break. If, however, there is any break in the graph, then function is not continuous at that point.

If for a value of k, the limit of f(x) does not exist i.e., if on the curve of f(x) a point is absent, the graph will be discontinuous i.e., not continuous.

A function f(x) is said to be continuous at x = a, $\lim_{x \to a} f(x)$ if exists in finite and is equal to f(a)

i.e., $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a)$

i.e., f (a + 0) = f (a – 0) = f (a) briefly or $\lim_{x \to 0} f(a + h) = f(a)$

Thus we are to find following three values :

(i) $\lim_{x \to a+} f(x)$, (ii) $\lim_{x \to a-} f(x)$ (iii) f(a)

If however all these values are equal, then f (x) is continuous at x = a, otherwise it is discontinuous.

Example 44 : Show that $f(x) = 3x^2 - x + 2$ is continuous at x = 1.

Solution:

Now, $\lim_{x\to 1+} f(x) = \lim_{x\to 1+}\left(3x^2 - x + 2\right) = \lim_{h\to 0}\left\{3(1+h)^2 - (1+h) + 2\right\}$

[Putting x = 1 + h as x → 1, h → 0]

$$= \lim_{h\to 0}\left(3h^2 + 5h + 4\right) = 3.0 + 5.0 + 4 = 4$$

and $\lim_{h\to 1-} f(x) = \lim_{x\to 1-}\left(3x^2 - x + 2\right) = \lim_{h\to 0}\left\{3(1-h)^2 - (1-h) + 2\right\}$

(x = 1 – h, h → 0 as x → 1)

$$\lim_{h\to 0}\left(3h^2 - 5h + 4\right) = 3.0 - 5.0 + 4 = 4$$

Again $f(1) = 3.1^2 - 1 + 2 = 4$

Thus we find that all the values are equal.

∴ f (x) is continuous at x = 1 ;

ε - δ Definition :

Again corresponding to definition of limit, we may define the continuity of a function as follows :

The functions f(x) is continuous at x = 1, if f (a) exists and for any pre-assigned positive quantity ε, however small we can determine a positive quantity δ, such that $|f(x) - f(a)| < \varepsilon$, for all values of x satisfying $|x - a| < \delta$.

Some Properties :

1. The sum or difference of two continuous functions is a continuous function

(i.e.,) $\lim_{x\to a}\{f(x) \pm \phi(x)\} = \lim_{x\to a} f(x) \pm \lim_{x\to a}\phi(x)$.

2. Product of two continuous functions is a continuous function.

3. Ratio of two continuous functions is a continuous function, provided the denominator is not zero.

Continuity in an Interval, at the End Points :

A function is said to be continuous over the interval (open or closed) including the end points if it is continuous at every point of the same interval.

Let c be any point in the interval (a, b) and if $\lim_{x\to c+} f(x) = \lim_{x\to c-} f(x) = f(c)$, then f(x) is continuous in the interval (a, b)

A function f(x) is said to continuous at the *left end* of an interval $a \le x \le b$ $\lim_{x\to a+} f(x) = f(a)$, if and at the *right end b* if $\lim_{x\to b-} f(x) = f(b)$.

Discontinuity at a Point : If at any point x = a in its domain, at least one of values $\lim_{x\to a+} f(x)$, $\lim_{x\to a-} f(x)$ and f(a) be different from the others, then f(x) fails to be continuous at that point, i.e., at x = a.

Example 45: Discuss the continuity of f (x) at x = 4, where

$f(x) = 2x + 1, \quad x \neq 4$

$\quad\;\; = 8 \qquad\quad x = 4.$

Here $\lim_{x \to 4+} f(x), \lim_{x \to 4+} (2x+1) = 2.4 + 1 = 9$

$\lim_{x \to 4-} f(x) = \lim_{x \to 4-} (2x+1) = 2.4 + 1 = 9$

f (4) = 8, which is different from the previous value.

$\therefore$ f (x) is not continuous at x = 4.

SOLVED EXAMPLES

Example 46: Show that f(X) = 2x + 3 is continuous at X = 1.

Solution :

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (2x+3) = 2.1 + 3 = 5.$

$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (2x+3) = 2.1 + 3 = 5.$ and at x = 1, f(1) = 2.1 + 3 = 5

$\therefore \lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x) = f(1) = 5$; $\therefore$ f(x) is continuous at x = 1.

Example 47: Discuss the continuity of f (x) = | x | at x = 0.

F (x) = | x | means f(x) = x for x > 0

= 0 for x = 0

= – x for x < 0.

Solution : $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x = 0$

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (-x) = 0$ and at x = 0, f(0) = 0

$\therefore \lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = f(0) = 0$; $\therefore$ f(x) is continous at x = 0.

Example 48: Show that $f(x) = \frac{1}{x-1}$ is discontinuous at x = 1.

Solution :

$f(1) = \frac{1}{1-1} = \frac{1}{0}$, undefined.

Now $\lim_{x \to 1+} f(x) = \lim_{h \to 0} \frac{1}{1+h-1} = \lim_{h \to 0} \frac{1}{h} = +\infty$ (Putting x = 1 + h, as x → 1, h → 0)

And $\lim_{x \to 1-} f(x) = \lim_{h \to 0} \frac{1}{1-h-1} = \lim_{h \to 0} \frac{1}{-h} = -\infty$ (Putting x = 1 – h, as x → 1, h → 0)

We find right hand limit is not equal to left hand limit.

$\therefore$ at x = 1, f (x) is discontinuous.

Example 49: Discuss the continuity at x = 2 where

f(x) = 4x + 8, $x \neq 2$

= 12 x = 2

Now $\lim_{x \to 2+} f(x) = \lim_{h \to 0} 4(2+h) + 8 = 16$ (Putting x = 2 + h as $x \to 2$, $h \to 0$)

And $\lim_{x \to 2-} f(x) = \lim_{h \to 0} 4(2-h) + 8 = 16.$ [for x = 2 – h, $h \to 0$, as $x \to 2$]

Again at x = 2, f(x) = f(2) = 12.

Three values are not equal. Hence f(x) is discontinuous at x = 2.

SELF EXAMINATION QUESTIONS

1. f (x) = x^2 + 1. Is the function continuous at x = 2? [Ans. Yes]

2. A functions is defined as follows f(x) = x^2, $x \neq 2$

= 2, x = 2

Is f(x) continuous at x = 2? [Ans. No]

3. Where are the following function discontinuous :

(i) $\frac{2}{x}$ (ii) $\frac{1}{x-1}$ (iii) $\frac{2x^2}{x-2}$ (iv) $\frac{1}{x^2-4x+3}$. [Ans. (i) 0, (ii) 1, (iii) 2, (iv) 1, 3]

4. Discuss the continuity of f(x) at x = c, where

$f(x) = \frac{x^2 - c^2}{x - c} x \neq c$

= 2c, x = c [Ans. Continuous]

5. (i) The function $f(x) = \frac{x^2 - 4}{x - 2}$ is undefined at x = 2. What value must be assigned to f(x), if f(x) is to be continuous at x = 2 ? Give reasons for your answer.

[Ans. $f(x) = \frac{x^2 - 4}{x - 2}$, $x \neq 2 = 4, x = 2$]

(ii) $f(x) = \frac{x^2 - 9}{x - 3}$ when $x \neq 3$. State the value of f(3) so that f(x) is continuous at x = 3.

[Ans. 6]

6. Is the function $f(x) = \frac{x^3 - 8}{x - 2}$ continuous at x = 2? If not what value must be assigned to f(2) to make f(x) continuous at x = 2. [Ans. 12]

7. (i) A function f(x) is defined as follows :

$f(x) = x + 1$, when $x \le 1$.

$= 3 - ax^2$ when $x > 1$.

For what value of a will f(x) be continuous? With this value of a draw the graph of f(x). [Ans.1]

(ii) Show that $f(x) = \frac{1}{x-2}$ is discontinuous at x = 2.

8. Show that f(x) is continuous at x = 0 and x = 1, where

$f(x) = x^2, \quad x \le 0$

$= x, \quad 0 < x < 1$

$= 2 - x, \quad x \ge 1.$

9. A function f(x) is defined as follows :

$f(x) = 1, \quad x > 0$

$= 0, \quad x = 0$

$= -1, \quad x < 0$

Show that it is discontinuous at x = 0.

OBJECTIVE QUESTIONS :

1. If f(x) = (x – 1) (x – 2) (x – 3), find f(0) [Ans. – 6]

2. If f(x) = x + | x |, find f(– 2) [Ans. 0]

3. If $f(x) = \frac{x^2 - 6x + 8}{x^2 + 8x + 12}$ find f(0) [Ans. $2/3$ $2/3$]

4. Given f(x) = x, $F(x) = \frac{x^2}{x,}$ is F(x) = f(x) always? [Ans. equal for $x \ne 0$]

5. If f(x)= 2 + x, x < 3 [Ans. 4]

= 7 – x, $x \ge 3$, find f(3)

6. If f(x)= (x – 2) (x – 3) (x + 4) find f(3) [Ans. 0]

7. Given $f(x) = \sqrt{x}$, for what value of x, f(x) is undefined? [Ans. – 1]

8. Find the range of the function $f(x) = \frac{x-2}{2-x}, x \ne 2$ [Ans. – 1]

9. If f(x) = x + $2x^3$, find – f(– x) [Ans. x + $2x^3$]

10. $f(x) = \frac{|x|}{x}$, $x \ne 0$ and is a real number, find $|f(c) - f(-c)|$. [Ans. 2]

11. If f(x) = e^{3x+4}, find f(1) + f(2) + f(5) [Ans. e^{36}]

12. If find $f(x)=\frac{1-x}{1+x}$, [Ans. $\frac{1}{2}$]

13. If f(x) = x + | x | are f(2) and f(– 2) equal? [Ans. no]

14. Evaluate : (i) $\lim_{x\to 1}\frac{x^2+1}{x+1}$ (ii) $\lim_{x\to 1}\frac{x^2-1}{x-1}$ (iii) $\lim_{x\to 2}\frac{|x-2|}{x-2}$ [Ans. (i) 1 (ii) 2 (ii) no exisstance]

15. Evaluate : (i) $\lim_{x\to 1}\frac{x^2+5x-6}{x-1}$ (ii) $\lim_{x\to 0}\frac{e^x-1}{x}$ [Ans. (i) 7 (ii) 1]

16. Evaluate : $\lim_{x\to -1}\frac{x^2+4x+3}{x^2-7x-8}$ [Ans. $\frac{-2}{9}$]

17. Find the value of $\lim_{x\to 0}\frac{1-\sqrt{1-x}}{x}$ [Ans. $\frac{1}{2}$]

18. If f(x) = x^2, evaluate : $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ [Ans. 2x]

19. Is the function f(x) = | x | continuous at x = 0 [Ans. Yes]

20. Evaluate : $\lim_{x\to\infty}\frac{6-5x^2}{4x+15x^2}$. [Ans. $\frac{1}{2}$]

21. A functions is defined as follows :

f(x) = 2x – 1, x < 3

= k, x = 3

= 8 – x, x > 3.

For what value of k, f(x) is continuous at x = 3? [Ans. 2]

22. A function is defined as follows :

f(x) = 1, x > 0

= 0, x = 0

= - 1 x < 0. Is the function is continuous at x = 0 [Ans. no]

3.4 DERIVATIVE

Introduction :

The idea of limit as discussed previously will now be extended at present in determining the derivative of a function f(x) with respect to x (the independent variable). For this let us know at first what the term 'increment' means.

Increment : By increment of a variable we mean the difference of initial value from the final value.

i.e., Increment = final value – initial value.

Let x change its value from 1 to 4, increment of x = 4 – 1 = 3.

Again if x changes from 1 to – 2, increment = – 2 – 1 = – 3.

(i.e., increment may be positive or negative).

Symbols : Increment of x will be denoted by h or, δx (delta x) or Δx (delta x) and that of y will be represented by k or, δy or, Δy.

If in y = f(x), the independent variable x changes to x + δx, then increment of x = x + δx – x = δx (≠ 0). So y = f(x) changes to y = f(x + δx).

∴ increment of y = f(x + δx) – f(x) [as y = f(x)]

Now the increment ratio $\frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x} = \frac{f(x+h)-f(x)}{h}$ [assuming δx = h]

If the ratio $\frac{\delta y}{\delta x}$ tends to a limit, as δx → 0 from either side, then this limit is known as the derivative of y [= f(x)] with respect to x.

Example : If y = 2x², then y + δy = 2 (x + δx)², δy = 2 (x + δx)² – 2x²

$$\therefore \frac{\delta y}{\delta x} = \frac{2(x+\delta x)^2 - 2x^2}{\delta x}$$

Again for $= \frac{1}{x^5}$, $\delta y = \frac{1}{(x+\delta x)^5} - \frac{1}{x^5}$

Definition : A function y = f(x) is said to be derivable at x if $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

or, $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$ or, $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

exists and equal to *l*. Now this limit *l* that exists is known as derivative (or differential co-efficient) of y [or f(x)] with respect to x.

Symbols : Derivative of y [= f(x)] w.r.t.x (with respect to x) is denoted by

$\frac{dy}{dx}$ or, f ′ (x), or, $\frac{d}{dx}$ f(x)

or, Dy or, D [f(x)] or, y_1

Now $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided this limit exists.

Note : $\frac{dy}{dx}$ does not mean the product of $\frac{d}{dx}$ with y. The notation $\frac{d}{dx}$ stands as a symbol to denote the operation of differentiation only. Read $\frac{dy}{dx}$ as 'dee y by dee x'.

SUMMARY :

The whole process for calculating f′ (x) or $\frac{dy}{dx}$ may be summed up in the following stages :

1. Let the independent variable x has an increment h and then find the new value of the function f (x + h).

2. Find f(x + h) – f(x).

3. Divide the above value by h i.e., find $\frac{f(x+h) - f(x)}{h}$.

4. Calculate $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$

SOME USEFUL DERIVATIVES :

1. $\frac{d}{dx} x^n = nx^{n-1}$

2. $\frac{d}{dx} . \frac{1}{x^n} = -\frac{n}{x^{n+1}}$.

3. $\frac{d}{dx} e^x = e^x$.

4. $\frac{d}{dx} a^x = a^x \log_e a$.

5. $\frac{d}{dx} \log_e x = \frac{1}{x}$.

6. $\frac{d}{dx} (\log_a x) = \frac{1}{x} \log_a e$.

7. $\frac{dc}{dx} = 0$ (c = constant)

8. $\frac{d}{dx} (u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

9. $\frac{d}{dx} (uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

10. $\frac{d}{dx} \frac{u}{v} = v \frac{\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

11. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Example 50 : $\frac{d}{dx}\left(x^4\right) = 4x^3$; $\frac{d}{dx}x = 1.x^{1-1} = 1x^0 = 1.1 = 1.$

$$\frac{d}{dx}\ \frac{1}{x}\ = \frac{d}{dx}\ x^{-1} = -1.x^{-1-1} = -1x^{-2} = \frac{-1}{x^2}$$

$$\frac{d}{dx}.\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}.x^{-1/2} = \frac{1}{2\sqrt{x}}\ ;$$

$$\frac{d}{dx}.\frac{1}{\sqrt{x}} = \frac{d}{dx}.x^{-\frac{1}{2}} = -\frac{1}{2}.x^{-3/2} = \frac{-1}{2x^{3/2}}\ ;\frac{d}{dx}\left(x\sqrt{x}\right)\ = \frac{d}{dx}.x^{3/2} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}\ ;$$

SELF EXAMINATION QUESTIONS

Examples 51: $\frac{d}{dx}(x^3 + x^2) = \frac{d}{dx}x^3 + \frac{d}{dx}x^2 = 3x^2 = 2x.$

Example 52 : $\frac{d}{dx}(x^2.e^x) = x^2\frac{d}{dx}e^x + e^x\frac{d}{dx}x^2 = x^2e^x + e^x.2x = x^2e^x + 2xe^x.$

Example 53: $\frac{d}{dx}(2x^4) = 2\frac{dx^4}{dx} = 2.4x^3 = 8x^3.$

Example 54 : If $y = \frac{x^2}{x+1}$. Let $y = \frac{u}{v}$ where u = x^2, $\frac{du}{dx} = \frac{d}{dx}\left(x^2\right) = 2x^{2-1} = 2x$

And $v = (x+1), \frac{dv}{dx} = \frac{d}{dx}(x+1) = \frac{d}{dx}x + \frac{d}{dx}.1 = 1 + 0 = 1$

Now $\frac{dy}{dx} = \frac{d}{dx}\ \frac{u}{v}\ = \frac{v\frac{du}{dx} - u.\frac{dv}{dx}}{v^2} = \frac{(x+1).2x - x^2.1}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}$

Example 55: Find $\frac{dy}{dx}$ of the following functions :

(i) $x^4 + 4x$, (ii) $3x^5 - 5x^3 + 110$, (iii) $-2 + (4/5)x^5 - (7/8)x^8$.

Solution:

(i) Let $y = x^4 + 4x$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^4 + 4x\right) = \frac{d}{dx}x^4 + \frac{d}{dx}\left(4x\right) = 4.x^{4-1} + 4\frac{d}{dx}.x = 4x^3 + 4.$$

(ii) Let $y = 3x^5 - 5x^3 + 110$

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^5 - 5x^3 + 110\right) = \frac{d}{dx}\left(3x^5\right) - \frac{d}{dx}\left(5x^3\right) + \frac{d}{dx}.110$$

$$= 3\frac{d}{dx}x^5 - 5\frac{d}{dx}x^3 + 0, \quad \text{(as 110 is a constant number)}$$

$= 3.5x^{5-1} + 5.3x^{3-1} = 15x^4 + 15x^2$.

(iii) Let $y = -2 + \frac{4}{5}x^5 - \frac{7}{8}x^8$

$$\frac{dy}{dx} = \frac{d}{dx}\ -2 + \frac{4}{5}x^5 - \frac{7}{8}x^8\ = \frac{d}{dx}(-2) + \frac{d}{dx}\ \frac{4}{5}x^5\ - \frac{d}{dx}\ \frac{7}{8}x^8$$

$$= 0 + \frac{4}{5}\frac{d}{dx}x^5 - \frac{7}{8}\frac{d}{dx}x^8, \quad \text{(as – 2 is a constant number)}$$

$$= +\frac{4}{5}.5x^{5-1} - \frac{7}{8}.8x^{8-1} = 4x^4 - 7x^7.$$

Example 56: If $s = ut + \frac{1}{2}ft^2$, find $\frac{ds}{dt}$ when t = 2.

$$\frac{ds}{dt} = \frac{d}{dt}\ ut + \frac{1}{2}ft^2\ = \frac{d}{dt}(ut) + \frac{d}{dt}\ \frac{1}{2}ft^2\ = u\frac{dt}{dt} + \frac{1}{2}f\frac{d}{dt}t^2$$

(here u, f, $\frac{1}{2}$ are constants & t is a variable, since we are to differentiate w.r.t. t)

$$= u.1 + \frac{1}{2}f.2t^{2-1} = u + \frac{1}{2}.2ft = u + ft$$

For $t = 2, \frac{ds}{dt} = u + 2f.$

REGARDING PRODUCT :

Example 57: Differentiate (x + 1) $(2x^3 — 21)$ with respect to x.

Let y = (x + 1) $(2x^3 – 21)$ = u. n where u = x + 1, n = $2x^3 – 21$

$$\frac{du}{dx} = \frac{d}{dx}(x+1) = \frac{dx}{dx} + \frac{d}{dx}.1 = 1 + 0 = 1$$

$$\frac{dn}{dx} = \frac{d}{dx}\left(2x^3 - 21\right) = 2\frac{d}{dx}.x^3 - \frac{d}{dx}(21) = 2.3x^{3-1} - 0 = 6x^2.$$

Now $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (x+1).\ 6x^2 + \left(2x^3 - 21\right).1$

$= 6x^3 + 6x^2 + 2x^3 – 21 = 8x^3 + 6x^2 – 21.$

Example 58: $y = x(x^2 - 1)(x^3 + 2)$, find $\frac{dy}{dx}$.

Let $y = uvw$, where $u = x$, $v = x^2 - 1$, $w = x^3 + 2$

$$\frac{du}{dx} = 1, \frac{dv}{dx} = 2x \text{ and } \frac{dw}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{d}{dx}(uvw) = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}$$

$= (x^2 - 1)(x^3 + 2).1 + x(x^3 + 2).2x + x(x^2 - 1).3x^2$

$= 6x^5 - 4x^3 + 6x^2$ (on simplification).

Example 59: If $y = 10^x x^{10}$, find $\frac{dy}{dx}$. Let $y = u.n$ where $u = 10^x$ and $v = x^{10}$

Now $\frac{du}{dx} = \frac{d}{dx}10^x = 10^x \log_e 10$;

Again $\frac{dv}{dx} = \frac{d}{dx}.x^{10} = 10.x^{10-1} = 10x^9$.

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(u.v.) = u\frac{dv}{dx} + v\frac{du}{dx} = 10^x \; 10x^9 + x^{10}.10^x \log_e 10 = 10^x(10x^9 + x^{10}\log_e 10)$

REGARDING DIVISION :

Example 60: If $y = \frac{x-1}{x+1}$, find $\frac{dy}{dx}$

Let $y = \frac{u}{v}$, where $u = x - 1$, $\frac{du}{dx} = \frac{d}{dx}(x-1) = \frac{dx}{dx} - \frac{d}{dx}.1 = 1 - 0 = 1$

$v = x + 1, \frac{dv}{dx} = \frac{d}{dx}(x+1) = \frac{dx}{dx} + \frac{d}{dx}1 = 1 + 0 = 1$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x+1).1-(x-1).1}{(x+1)^2} = \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Example 61: If $y = \frac{(x+1)(2x^2-1)}{x^2+1}$, find $\frac{dy}{dx}$.

Let $y = \frac{u}{v}$, where $u = (x + 1)(2x^2 - 1) = 2x^3 + 2x^2 - x - 1$.

$\frac{du}{dx} = 6x^2 + 4x - 1$ and $v = x^2 + 1$; $\frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x^2+1)(6x^2+4x-1)-(x+1)(2x^2-1).2x}{(x^2+1)^2} = \frac{2x^4+3x^3+2x-1}{(x^2+1)^2}$$

SELF EXAMINATION QUESTIONS

Part A-regarding sum and difference of functions :

Differentiate the following functions with respect to x (or independent variable) : (a, b, c, m, n, are constants):

1. (i) $x^7 + 7$ (ii) $7x^7 + 7$ [Ans. (i) $7x^6$, (ii) 49^6]

2. (i) $ax + b + c$ (ii) $ax^2 - bx$ (iii) $\frac{a}{4}.x^4 + \frac{b}{2}x^2 + c$ (iv) $x^5 + bx^3 - cx$.

[Ans. (i) a (ii) $2ax - b$ (iii) $ax^3 + bx$ (iv) $5x^4 + 3bx^2 - c$]

3. (i) $x^m + mx$ (ii) $x^m + x^{m-1}$

[Ans. (i) $mx^{m-1} + m$ (ii) $mx^{m-1} + (m-1)x^{m-2}$]

4. (i) $\sqrt{x} + \sqrt{4}$ (ii) $2\sqrt{x} + 3x$ (iii) $\left(\sqrt{x} + 1\right)^2$ (iv) $4\left(\sqrt{x} + x^2\right)$

[Ans. (i) $\frac{1}{2\sqrt{x}}$ (ii) $\frac{1}{\sqrt{x}} + 3$ (iii) $1 + \frac{1}{\sqrt{x}}$ (iv) $\frac{2}{\sqrt{x}} + 8x$]

5. (i) $2x^2 + \frac{2}{x}$ (ii) ${x + \frac{1}{x}}^{\,2}$ (iii) $\sqrt{x} + \frac{1}{\sqrt{x}}$ (iv) ${\sqrt{x} - \frac{1}{\sqrt{x}}}^{\,2}$

[Ans. (i) $4x - \frac{2}{x^2}$ (ii) $2x - \frac{2}{x^3}$ (iii) $\frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$ (iv) $1 - \frac{1}{x^2}$]

6. (i) $\frac{2x^3 + 3x^2 + 4}{x^2}$ (ii) $\frac{x^3 + 2x^2 - 1}{\sqrt{x}}$ [Ans.(i) $2 - \frac{8}{x^3}$ (ii) $\frac{5}{2}x^{3/2} + 3\sqrt{x} + \frac{1}{2x^{3/2}}$]

7. If $y = 4x^6 + 2x^3 + x - 1000$, find $\frac{dy}{dx}$ when x = - 1. [Ans. –17]

8. If $y = 4x^3 - 15x^2 + 12x + 3 = 0$; find the value of x for which $\frac{dy}{dx} = 0$. [Ans. $\frac{1}{2}$, 2]

9. If y = x . (i) prove that $x\frac{dy}{dx} = 3y$, (ii) find the value of ${1 + \frac{dy}{dx}}^{\,2}$ [Ans. 10]

10. Given $\frac{6x^5 - 3x^3(\log x + 2) + 5}{x^3}$; find $\frac{dy}{dx}$ [Ans. $12x - \frac{15}{x^4} - \frac{3}{x}$]]

Part B Regarding product functions :

Differentiate the following with respect to x

11. (i) $x (x + 1)$ (ii) $x^2 (x - 1)$ [Ans. (i) $2x + 1$ (ii) $3x^2 - 2x$]

12. (i) $(x + 1) (x + 2)$ (ii) $(x^2 + 1) x^2$ [Ans. (i) $2x + 3$ (ii) $4x^3 + 2x$]

14. (i) $(x - 1)^2 (x + 2)$ (ii) $(x^2 + 1) (x - 2)^3$. [Ans. (i) $3x^2 - 3$ (ii) $(x - 2)^2 (5x^2 - 4x + 3)$]

15. (i) $x. e^x$ (ii) $x^{10}e^x$ (iii) $e^x. \log x$ (iv) $2e^x (\log x + 4)$.

[Ans. (i) $x.e^x + e^x$ (ii)$x^{l0}e^x$+ lOxV (iii) ex.$\frac{1}{x}$ + $e^x\log x$ (iv) $2e^x.\frac{1}{x}$ + $2e^x (\log x + 4)$]

16. (i) $(x+ l)(x + 2)(x + 4)$ (ii) $(x^2+ 1) (x+ 2^2) (x^2-2)$ [Ans. (i) $3x^2 + 14x + 14$ (ii) $5x^4 + 16x^3 - 3x^2 - 8x - 2$]

Part C Regarding division of functions :

Differentiate the following with respect to x.

17. (i) $\frac{x}{1+x^2}$ (ii) $\frac{x^2}{1+x}$ (iii) $\frac{x+1}{x+2}$ (iv) $\frac{x+4}{x^2+2}$

[Ans. (i) $\frac{1-x^2}{(1+x^2)^2}$ (ii) $\frac{2x+x^2}{(1+x)^2}$ (iii) $\frac{1}{(x+2)^2}$ (iv) $\frac{-(x^2+8x-2)}{(x^2+2)^2}$

18. (i) $\frac{x^2+1}{x-1}$ (ii) $\frac{x+1}{x^3-1}$ (iii) $\frac{(2-5x)^2}{(x^3-1)}$ (iv) $\frac{x^4-1}{x^4+1}$

[Ans. (i) $\frac{x^2-2x-1}{(x-1)^2}$ (ii) $\frac{-(2x^3+3x^2+1)}{(x^3-1)^2}$

(iii) $\frac{-25x^4+40x^3-12x^2-50x+20}{(x^3-1)^2}$ (iv) $\frac{8x^4}{(x^4+1)^2}$]

19. (i) $\frac{x^2}{\log x}$ (ii) $\frac{\log x}{x^2}$ (iii) $\frac{e^x}{x}$ (iv) $\frac{x^2}{e^x}$

[Ans. (i) $\frac{2x\log x - x}{(\log x)^2}$ (ii) $\frac{x-2x\log x}{x^4}$ (iii) $\frac{e^x(x-1)}{x^2}$ (iv) $\frac{xe^x(2-x)}{e^{2x}}$]

3.4.1 DERIVATIVE OF FUNCTION OF A FUNCTION

A variable y may be a function of a second variable z, which again may be a function of a third variable x.

i.e., $y = z^2 + 3$, and $z = 2x + 1$

Here y is a function of z and z again a function of x. Ultimately y is seen to depend on x, so y is called the *function of another function*.

Symbolically, if $y = f (z)$, $z = \phi (x)$ then $y = f \{\phi (x)\}$

Theorem. If y = f(z) and z = φ (x) then $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}$ (proof is not shown at present)

Corr. If u = f(v), v = φ (w), w = ψ (x) then $\frac{du}{dx} = \frac{du}{dv}.\frac{dv}{dw}.\frac{dw}{dx}$

Example 62 : To find $\frac{dy}{dx}$ for $y = 2z^2 + 1$, z = 4x – 2

Solution:

Now $\frac{dy}{dz} = 4z$ and $\frac{dz}{dx} = 4$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = 4z.\ 4 = 16z = 16\,(4x - 2) = 64x - 32.$$

Rule 1. If y = ax + b, to find $\frac{dy}{dx}$. Let y = z, and z = ax + b.

So y = f (z) and z = f (x)

∴ $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = 1.\ (a.\ 1 + 0) = a$

Rule. 2. If $y = (ax + b)^n$, to find $\frac{dy}{dx}$ Let $y = z^n$ and z = ax + b

Now $\frac{dy}{dz} = n.z^{n-1}$ and $\frac{dz}{dx} = a$

∴ $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = nz^{n-1}\ a = na\,(ax + b)^{n-1}$

Example 63 : If $\{y = (2x + 5)^4]$ Let $y = z^4$, where z = 2x + 5.

Now $\frac{dy}{dz} = 4z^3, \frac{dz}{dx} = 2$

Now $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = 4z^3.\ 2 = 4.2(2x+5)^3 = 8(2x+5)^3$

Rule 3. If y = log u (u is a function of x), then to find $\frac{dy}{dx}$; $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

Example 64 : $y = \log(4x)$, find $\frac{dy}{dx}$

Solution: Let $y = \log u$, where $u = 4x$, $\frac{du}{dx} = 4$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}(\log u).\ 4 = \frac{1}{u} \cdot 4 = 4.\frac{1}{4x} = \frac{1}{x}.$$

Example 65 : $y = \log\left(1+\sqrt{x}\right)$, find $\frac{dy}{dx}$

Solution: Let $y = \log u$, where $u = 1+\sqrt{x}$

$$\frac{du}{dx} = 0 + \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}};\ \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{u}.\frac{1}{2\sqrt{x}} = \frac{1}{\left(1+\sqrt{x}\right).\ 2\sqrt{x}}$$

Example 66: Find $\frac{dy}{dx}$ if $y = (2x-5)^6$.

Solution: Let $y = z^6$, where $z = 2x - 5$, $\frac{dz}{dx} = 2$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{d}{dz}\left(z^6\right).2 = 6z^5.2 = 12z^5 = 12\ (2x-5)^5$$

Example 67: If $y = \sqrt{x^2+7}$, find $\frac{dy}{dx}$.

Solution: Let $y = \sqrt{z}$, where $z = x^2+7$, $\frac{dz}{dx} = 2x$.

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{d}{dz}.\sqrt{z}.\ 2x = \frac{1}{2\sqrt{z}}.\ 2x = \frac{x}{\sqrt{z}} = \frac{x}{\sqrt{x^2+7}}$$

Example 68: If $y = \left(x^3+2x^2+5x\right)^{-3}$, find $\frac{dy}{dx}$

Solution: Let $y = u^{-3}$, where $u = x^3 + 2x^2 + 5x$,

$$\frac{du}{dx} = 3x^2+4x+5 \qquad \frac{dy}{du} = -3u^{-4}$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{d}{du}.u^{-3}.\ \left(3x^2+4x+5\right)$$

$$= -3.u^{-4}.\ (3x^2 + 4x + 5) = -3\ (3x^2 + 4x + 5).\ (x^3 + 2x^2 + 5x)^{-4}$$

Example 69: Given $y=\sqrt{\frac{2x+1}{x+2}}$, find $\frac{dy}{dx}$.

Solution:

Let $y=\sqrt{u}$, where $u=\frac{2x+1}{x+2}$, $\frac{du}{dx}=\frac{(x+2)\frac{d}{dx}(2x+1)-(2x+1)\frac{d}{dx}(x+2)}{(x+2)^2}$

$$\frac{du}{dx}=\frac{(x+2).2-(2x+1).1}{(x+2)^2}=\frac{3}{(x+2)^2}$$

$$\frac{dy}{du}=\frac{1}{2\sqrt{u}}=\frac{1}{2}.\sqrt{\frac{x+2}{2x+1}}$$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{2}\sqrt{\frac{x+2}{2x+1}}.\frac{3}{(x+2)^2}=\frac{3}{2\sqrt{2x+1}.(x+2)^{3/2}}$$

Example 70: If y = log log log x^2, find $\frac{dy}{dx}$

Solution :

Let y = log u where u = log v and v = log x^2 = 2 logx.

$$\frac{dy}{du}=\frac{1}{u}=\frac{1}{\log v}=\frac{1}{\log\log x^2}.$$

$$\frac{du}{dv}=\frac{1}{v}=\frac{1}{\log x^2}=\frac{1}{2\log x}.$$

$$\frac{dv}{dx}=\frac{2}{x}$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}=\frac{1}{\log\log x^2}.\frac{1}{2\log x}.\frac{2}{x}=\frac{1}{x\log x\log\log x^2}$$

Example 71: Differentiate x^5 w.r.t.x^2

Solution:

Let y = x^5, z = x^2

$$\frac{dy}{dx}=5x^4,\frac{dz}{dx}=\frac{d}{dx}x^2=2x, \text{ so that } \frac{dx}{dz}=\frac{1}{2x}.$$

Now $\quad \frac{dy}{dz}=\frac{dy}{dx}.\frac{dx}{dz}=5x^4.\frac{1}{2x}=\frac{5}{2}x^3.$

Example 72: $y = \log_e\left(x+\sqrt{x^2+a^2}\right)$, find $\frac{dy}{dx}$.

Solution:

Let $y = \log u$ where $u = x + \sqrt{x^2+a^2}$

$$\frac{du}{dx}=\frac{d}{dx}x+\frac{d}{dx}\left(x^2+a^2\right)^{1/2} \quad \text{or} \quad \frac{du}{dx}=1+\frac{1}{2\sqrt{x^2+a^2}}.2x$$

$$\text{Now } \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{u}.\ 1+\frac{x}{\sqrt{x^2+a^2}} = \frac{1}{x+\sqrt{x^2+a^2}}.\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}=\frac{1}{\sqrt{x^2+a^2}}.$$

SELF EXAMINATION QUESTIONS

Differentiate the following functions w.r.t.x :

1. $(x^2 + 5)^2$. [Ans. $4x(x^2 + 5)$]

2. (i) $(ax + b)^5$. [Ans. $5a(ax + b)^4$] (ii) $(1 - 5x)^6$. [Ans. $-30(1 - 5x)^5$]

(iii) $(3 - 5x)^{3/2}$. [Ans. $-\frac{15}{2}\sqrt{3-5x}$]

3. $(x^3 + 3x)^4$ [Ans. $12(x^2 + 1)(x^3 + 3x)^3$]

4. $\sqrt{3x^2+7}$ [Ans. $\frac{3x}{\sqrt{3x^2-7}}$]

5. $(2x^2 + 5x - 7)^{-2}$ [Ans. $-2(4x + 5)(2x^2 + 5x - 7)^{-3}$]

6. $x^3\sqrt{1-x^2}$. [Ans. $3x^2\sqrt{1-x^2}-\frac{x^4}{\sqrt{1-x^2}}$]

7. $\sqrt{\frac{x^2-1}{x^2+1}}$ [Ans. $\frac{2x}{\sqrt{\left(x^2-1\right)}\left(x^2+1\right)^{3/2}}$]

8. e^{4x} [Ans. $4e^{4x}$]

9. (i) e^{3x^2+4x-7} [Ans. $(6x + 4)e^{3x^2+4x-7}$]

(ii) e^{3x^2-6x+2} [Ans. $6(x - 1)e^{3x^2-6x+2}$]

10. $\log(x^2 + 2x + 5)$. [Ans. $\frac{2(x+1)}{x^2+2x+5}$]

11. $\log\left(\sqrt{x+1}-\sqrt{x-1}\right)$. [Ans. $\frac{-1}{2\sqrt{x^2-1}}$]

12. log log log x^2. [Ans. $\frac{1}{\log\log x^2}\cdot\frac{1}{\log x^2}\cdot\frac{2}{x}$]

13. If $y=\sqrt{1+x^2}$, prove that $y\frac{dy}{dx}=x$.

14. Differentiate x^6 w.r.t.x^4. [Ans. $\frac{3}{2}x^2$]

15. Differentiate x^5 w.r.t.x^2 [Ans. $\frac{5}{2}x^3$]

3.4.2 DERIVATIVE OF IMPLICIT FUNCTION

If f(x, y) = 0 defines y as a derivable function of x, then differentiate each term w.r.t.x. The idea will be clear from the given example.

Example 73 : Find $\frac{dy}{dx}$, if $3x^4 - x^2y + 2y^3 = 0$

Solution:

Differentiating each term of the functions w.r.t.x we get, $3.4x^3 - \quad x^2\frac{dy}{dx}+2xy \quad +6y^2\frac{dy}{dx}=0$

or, $12x^3 - x^2\frac{dy}{dx} - 2xy + 6y^2\frac{dy}{dx} = 0$

or, $\left(6y^2 - x^2\right)\frac{dy}{dx} = 2xy - 12x^3$ or, $\frac{dy}{dx} = \frac{2xy-12x^3}{6y^2-x^2}$.

3.4.3 DERIVATIVE OF PARAMETRIC FUNCTION

Each of variable x and y can be expressed in terms of a third variable (known as parametric function). For example. $X = f_1(t), y = f_2(t)$. Now to find $\frac{dy}{dx}$ we are to find $\frac{dy}{dt}$ and $\frac{dx}{dt}$ so that $\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}=\frac{dy/dt}{dx/dt}, \frac{dx}{dt}\neq 0$.

Example 74 : Find $\frac{dy}{dx}$ when x = 4t, y = $2t^2$

Solution:

$$\frac{dx}{dt}=4, \frac{dy}{dt}=4t, \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{4t}{4}=t$$

Example 75 : Find $\frac{dy}{dx}$, when $x=\frac{3\,at}{1+t^3}$, $y=\frac{3at^2}{1+t^3}$

Solution:

$$\frac{dx}{dt}=\frac{\left(1+t^3\right).3a-3at\left(3t^2\right)}{\left(1+t^3\right)^2}=\frac{3a\left(1-2t^2\right)}{\left(1+t^3\right)^2}$$

$$\frac{dy}{dt}=\frac{(1+t^3)\,6at-3at^2(3t^2)}{(1+t^3)^2}=\frac{3at\,(2-t^3)}{(1+t^3)^2}$$

$$\frac{dy}{dx}=\frac{(1+t^3)\,6at-3at^2(3t^2)}{(1+t^3)^2}=\frac{3at\,(2-t^3)}{(1+t^3)^2}$$

SELF EXAMINATION QUESTIONS

Find derivative of the following functions w.r.t. x

1. $2x^2 + 5xy + 3y^2 = 1$ [Ans. $\frac{-(4x+5y)}{(5x+6y)}$]

2. $x^3 + y^3 = a^3$ [Ans. $\frac{-x^2}{y^2}$]

3. $x^3 + y^3 = 3axy$ [Ans. $\frac{ay-x^2}{y^2-ax}$]

4. $x = at^2,\ y = 2at$ [Ans. $\frac{1}{t}$]

5. $x = at,\ y = \frac{a}{t}$ [Ans. $-\frac{1}{t^2}$]

6. $2x = t^2,\ 3y = t^3$ [Ans. t]

7. $x = 5t - t^3,\ y = t^2 + 4,$ at $t = 1$ [Ans. 1]

8. $x = \frac{1}{t}, y = 4t,$ at $t = -2$ [Ans. – 16]

9. $x = 2at, y = at^2$ [Ans. t]

10. $2x^2 + 3xy + y^2 = 4$ at the point (0, 2) [Ans. $-\frac{3}{2}$]

3.4.4 SECOND ORDER DERIVATIVE

Introduction :

We have seen that the first order derivative of a function of x, say f(x), may also be a function of x. This new function of x also may have a derivative w.r.t.x which is known as second order derivative of f(x) i.e. second order derivative is the derivative of first order.

Similarly the derivative of the second order derivative is known as third order derivative and so on up to *n*th order.

Symbols :

For the function y = f(x), is first order derivative w.r.t.x denoted by $\frac{dy}{dx}$ or f′ (x) or y_1 as discussed before.

Now the second order derivative of y = f(x) is expressed as $\frac{d^2y}{dx^2}$ or f ″ (x) or y_2. The notation $\frac{d^2y}{dx^2}$ is read as "*dee two y by dee x squared*".

Example 76 : If $y = x^4$, find $\frac{d^2y}{dx^2}$.

Solution:

$$y = x^4, \frac{dy}{dx} = 4x^3, \text{ again} \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}4x^3 = 4.3x^{3-1} = 12x^2.$$

Note : To find $\frac{d^3y}{dx^3}$; $\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left(12x^2\right) = 12.\frac{d}{dx}x^2$

$= 12.2x^{2-1} = 24x$ i.e., third order derivative is 24x.

Example 77 : Find $\frac{d^2y}{dx^2}$ if $y = \frac{\log x}{x}$.

Solution:

$$\frac{dy}{dx} = y_1 = \frac{x.\frac{1}{x} - \log x.1}{x^2} = \frac{1-\log x}{x^2};$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\frac{1-\log x}{x^2} = \frac{x^2 \quad -\frac{1}{x} \quad -(1-\log x).\ 2x}{x^3}$$

$$= \frac{-x - 2x(1-\log x)}{x^3} = \frac{-1-2\ (1-\log x)}{x^2} = \frac{2\log x - 3}{x^2}.$$

Example 78 : If $y = 5^x$, find $\frac{d^2y}{dx^2}$

y =5x or, log y = x log 5

or, $\frac{1}{y}.\frac{dy}{dx} = \log 5$ or $\frac{dy}{dx} = y.\log 5$ or $\frac{dy}{dx} = 5x.\log 5$ .... (i)

Again $\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}(5x.\log 5)$ or $\frac{d^2y}{dx^2} = 5^x.\frac{d}{dx}\ \log 5 + \log 5 \cdot \frac{d}{dx} \cdot 5^x$

or, $\frac{d^2y}{dx^2} = 5^x.0 + \log 5.\log 5.5^x$ [from (i)]

$= (\log 5)^2.5^x$

Example 79 :

If y = x log x find $\frac{d^2y}{dx^2}$

Solution: $Y = x \log x$

D.w.r. to x

$$\frac{dy}{dx} = x \times \frac{1}{x} + \log x.1$$

$$\frac{dy}{dx} = 1 + \log x$$

D.w.r. to x

$$\frac{d^2y}{dx^2} = 0 + \frac{1}{x}$$

$$= \frac{1}{x}$$

Example 80 :

If $y = ax^2 + bx + C$ find $\frac{d^2y}{dx^2}$

Solution: $y = ax^2 + bx + C$

D.w.r. to x

$$\frac{dy}{dx} = 2ax + b$$

D.w.r. to x

$$\frac{d^2y}{dx^2} = 2a$$

Example 81 :

If $y = x^2 e^x$, Prove $\frac{d^2y}{dx^2} = (x^2 + 4x + 2)e^x$

Solution:

$y = x^2 e^x$

D.w.r. to x

$$\frac{dy}{dx} = x^2.e^x + e^2.2x$$

$= (x^2 + 2x)e^x$

D.w.r. to x

$$\frac{d^2y}{dx^2} = (x^2 + 2x)\ e^x + e^x\ (2x + 2)$$

$$= (x^2 + 2x + 2x + 2)e^x$$

$$= (x^2 + 4x +2)e^x$$

FOR IMPLICIT FUNCTION AND PARAMETRIC FORMS :

Example 82 : For $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$ find $\frac{d^2y}{dx^2}$

Solution:

Diff. Both sides w.r.t. x we get $\frac{2x}{a^2}+\frac{2y.y_1}{b^2}=0$ or $\frac{2x}{a^2}+\frac{2y}{b^2}.y_1=0$ or $y_1=-\frac{b^2}{a^2}.\frac{x}{y}$

$$\frac{d^2y}{dx^2}= -\frac{b^2}{a^2}.\frac{y.1-x.y_1}{y^2}= -\frac{b^2}{a^2}\frac{y+x\ \frac{b^2}{a^2}.\frac{x}{y}}{y^2}$$ [putting the value of y_1]

$$= -\frac{b^2}{a^2}.\frac{(a^2y^2+b^2x^2)}{a^2y^3}= -\frac{b^2}{a^2}.\frac{a^2b^2}{a^2y^3}$$ [from the given expression]

$$=\frac{b^4}{a^2y^3}.$$

Example 83 : If y = log x

Solution:

$$y_1=\frac{dy}{dx}=\frac{1}{x},\ y_2=\frac{d^{2y}}{dx^2}=-\frac{1}{x^2}.$$

Example 84:

If $y = t^2 + t^3$, $x = t - t^4$, find $\frac{d^2y}{dx^2}$.

Solution:

$$\frac{dy}{dt}=2t+3t^2,\ \frac{dx}{dt}=1-4t^3$$

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{2t+3t^2}{1-4t^3}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{d}{dt}\frac{dy}{dx}.\frac{dt}{dx}=\frac{d}{dt}\frac{dy}{dx}\Big/\frac{dx}{dt}=\frac{d}{dt}\frac{2t+3t^2}{1-4t^3}\Big/(1-4t^3)$$

$$=\frac{(1-4t^3)(2+6t)-(2t+3t^2)(-12t^2)}{(1-4t^3)^2.(1-4t^2)}=\frac{12t^4+16t^3+6t+2}{(1-4t^3)^2}$$

SELF EXAMINATION QUESTIONS

Find $\frac{d^2y}{dx^2}$ in the following cases :

1. (i) $y = 2x$. (ii) $y = 2x^4$. (iii) $y = 5x^3-3x^2$. (iv) $y = (2 + 3x)^4$.

[Ans. (i) 0 (ii) $24x^2$ (iii) $30x - 6$ (iv) $108 (2 + 3x)^2$]

2. (i) $y = x^4 e^{2x}$ (ii) $y = x (1 - x)^2$ (iii) $y = x^4 \log x$.

[Ans. (i) $4x^2 e^{2x} (x^2 + 4x + 3)$ (ii) $6x - 4$ (iii) $7x^2 + 12x^3.\log x$]

(iii) $7x^2 + 12x^3 .loc^x$)

3. (i) $y = \frac{\log x}{x}$ (ii) $y = \frac{1+x}{1-x}$ [Ans. (i) $\frac{2\log-3}{x^3}$ (ii) $\frac{4}{(1-x)^3}$]

4. (i) $x = at^2$, $y = 2at$, (ii) $x = \frac{t^2}{1+t}, y = \frac{t}{1+t}$ (iii) $x = \frac{1-t}{1+t}, y = \frac{2t}{1+t}$

[Ans. (i) $\frac{-1}{2at^3}$ (ii) $\frac{-2(t+1)^3}{(t^2+2t)^3}$ (iii) 0]

5. If $y = x^3 - 9x^2 + 9x$ find $\frac{d^2y}{dx^2}$ for x= 1, x = 3. [Ans. – 12, 0]

3.4.5 PARTIAL DERIVATIVE

We know that area of a rectangle is the product of its length and breadth, i.e., area = $l \times b$. Now if the length increases (when breadth is constant), area increases. If again breadth decreases (taking length as constant), area decreases. So here area is a function of two independent variables, (i.e., length and breadth).

Again we know that area of a triangle is $\frac{1}{2}$ base × altitude, i.e., area is a function of base and altitude, (i.e., two variables).

If u is a function of two independent variables x and y, then we may write $u = f(x, y)$.

The result obtained in differentiating $u = f(x, y)$ w.r.t.x, treating y as a constant, is called the partial derivative of u w.r.t.x and is denoted by any one of $\frac{\partial u}{\partial x}.\frac{\partial f}{\partial x}$, $f_x(x, y)$, u_x, f_x where

$$\frac{\partial f}{\partial x} \text{ or } f_x = \lim_{\partial x \to 0} \frac{f(x+\partial x, y) - f(x, y)}{\partial x}, \text{ if it exists}$$

Similarly, the partial derivative of $u = f(x, y)$, w.r.t. y (treating x as a constant) is the result in differentiating $u = f(x, y)$, w.r.t. y and is denoted by $\frac{\partial u}{\partial y}, \frac{\partial f}{\partial y}$ or f_y where $\frac{\partial f}{\partial y}$ or $f_y = \lim_{\partial y \to 0} \frac{f(x, y+\partial y) - f(x, y)}{\partial y}$ provided the limit exists.

Note : The curl is used for partial derivative in order to make different form of symbol d of ordinary derivative.

Example 85 : $u = (x + y)^2 = x^2 + 2xy + y^2$, $\frac{\partial u}{\partial x} = 2x + 2y, \frac{\partial u}{\partial y} = 2x + 2y.$

Function of three variables : A function may be of three variables also, i.e., $u = f(x, y, z)$. Now the partial derivative of u w.r.t. x is the derivative of *u w.r.t.x (treating y and z as constant)*.

Example 86 : $u = x^2 + y^2 + z^2$; $\frac{\partial u}{\partial x} = 2x, \ \frac{\partial u}{\partial y} = 2y, \frac{\partial u}{\partial z} = 2z.$

Partial derivative of higher order

Partial derivative of higher order is obtained by usual method of derivative.

For the function $u = f(x, y)$, we have following four partial derivative of second order :

(i) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \ \frac{\partial u}{\partial x} = f_{xx}$ (ii) $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \ \frac{\partial u}{\partial y} = f_{yy}$

(iii) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \ \frac{\partial u}{\partial y} = f_{yx}$ (iv) $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \ \frac{\partial u}{\partial x} = f_{xy}.$

Example 87 : $u = 2x^2 - 4xy + 3y^2$ find $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial x \partial y}$

$$\frac{\partial u}{\partial x} = 4x - 4y, \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \ \frac{\partial u}{\partial x} \ = \frac{\partial}{\partial x}(4x - 4y) = 4.$$

Again $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \ \frac{\partial u}{\partial x} \ = \frac{\partial}{\partial y}(4x - 4y) = -4.$

$$\frac{\partial u}{\partial y} = -4x + 6y, \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \ \frac{\partial u}{\partial y} \ = \frac{\partial}{\partial x}(-4x + 6y) = -4$$

Note : (i) $\frac{\partial^2 u}{\partial y \partial x}$ means partial derivative of $\frac{\partial u}{\partial x}$ w.r.t.y.

(ii) $\frac{\partial^2 u}{\partial x \partial y}$ means partial derivative of $\frac{\partial u}{\partial y}$ w.r.t.x.

(iii) We get $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ (from the above result) i.e., $u_{xy} = u_{yx}$.

Example 88: Find partial derivative of first order of the function $x^2 + 4xy + y^2$

Solution:

Let $u = f(x, y) = x^2 + 4xy + y^2$.

Now differentiating u w.r.t.x (treating y as a constant) we find

$$f_y \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + 4xy + 3y^2) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(4xy) + \frac{\partial}{\partial x}(y^2) = 2x + 4y + 0 = 2x + 4y.$$

Again (treating x as a constant)

$$f_y \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 + 4xy + 3y^2) = \frac{\partial}{\partial y} + (x^2) + \frac{\partial}{\partial y}(4xy) + \frac{\partial}{\partial y}(y^2)$$

= 0 + 4x + 2y = 4x + 2y.

Example 89: If $u = x^4 y^3 z^2 + 4x + 3y + 2z$ find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.

Solution:

$$\frac{\partial u}{\partial x} = 4x^3y^3z^2 + 4, \quad \frac{\partial u}{\partial y} = 3y^2x^4z^2 + 3, \quad \frac{\partial u}{\partial z} = 2zx^4y^3 + 2.$$

Example 90: If $u = \log(x^2 + y^2)$; find $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$.

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2}.2x, \quad \frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2).2 - 2x(2x)}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2)2 - 2y(2y)}{(x^2 + y^2)^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

Example 91:

For $f(x, y) = 3x^2 - 2x + 5$, find f_x and f_y

Solution:

$f_x = 6x - 2$ and $f_y = 0$.

Example 92 :

If $f(x, y) = \frac{x}{y^3} - \frac{y}{x^3}$, find f_x and f_y

Solution:

$$f_x = \frac{1}{y^3} + \frac{3y}{x^4} \qquad f_y = -\frac{3x}{y^4} - \frac{1}{x^3}$$

Example 93:

Find the first partial derivatives of $f(x, y, z) = xy^2z^3$

Solution:

$f_x = 2 - u$ $\qquad$ $f_y = z + 2uy$ $\qquad$ $f_x = 3xy^2z^2$

Example 94 :

Find the first partial derivatives of f(x, y, z, u, v) = 2x + yz – vx + vy².

Solution:

$F_x = 2 - u$ $f_y = z + 2vy$ $f_z = y$ $f_u = -x$ $f_v = y^2$

Example 95 :

If $f(x, y) = 3x^2y - 2x^3 + 5y^2$, find $f_x(1, 2)$ and $f_y(1, 2)$.

Solution:

$F_x = 6xy - 6x^2$.

Hence, $f_x(1, 2) = 12 - 6 = 6$. $F_y = 3x^2 + 10y$.

Hence, $f_y(1, 2) = 3 + 20 = 23$.

Example 96 :

For $f(x, y) = 3x^2y - 2xy + 5y^2$, verify that $f_{xy} = f_{yx}$.

Solution:

$f_x = 6xy - 2y$, $f_{xy} = 6x - 2$. $F_y = 3x^2 - 2x + 10y$, $f_{yx} = 6x - 2$.

Example 97 :

If $f(x, y) = 3x^2 - 2xy + 5y^3$, verify that $f_{xy} = f_{yx}$.

Solution:

$f_x = 6x - 2y$ $f_{xy} = -2$ $f_y = -2x + 15y^2$ $f_{yx} = -2$

Example 98:

For $f(x, y) = 3x^4 - 2x^3y^2 + 7y$, find f_{xx}, f_{xy}, f_{yx}, and f_{yy}.

Solution:

$F_x = 12x^3 - 6x^2y^2$, $f_y = -4x^3y + 7$, $f_{xx} = 36x^2 - 12xy^2$, $f_{yy} = -4x^3$, $f_{xy} = -12x^2y$, $f_{yx} = -12x^2y$.

Example 99 :

If $f(x, y, z) = x^2y + y^2z - 2xz$, find f_{xy}, f_{yx}, f_{xz}, f_{zx}, f_{yz}, f_{zy}.

Solution:

$f_x = 2xy - 2x$, $f_y = x^2 + 2yz$, $f_z = y^2 - 2x$. $f_{xy} = 2x$, $f_{yx} = 2x$, $f_{xz} = -2$, $f_{zx} = -2$, $f_{yx} = 2y$, $f_{zy} = 2y$.

SELF EXAMINATION QUESTION

1. f_x, f_y for the following f(x, y) :

(i) $(x-y)^2$ (ii) $x^3 + 3xy + y^3$ (iii) $\sqrt{x^2+y^2}$ (iv) $\frac{1}{\sqrt{x^2+y^2}}$

(v) $\frac{x+y}{(x-y)}$ (vi) $\frac{x^3+y^3}{x+y}$ (vii) e^{xy} (viii) log (xy).

[Ans. (i) 2 (x – y) ; – 2 (x – y) (ii) $3x^2 + 3y : 3x + 3y^2$

(iii) $\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}$ (iv) $-x.\left(x^2+y^2\right)^{-3/2}; -y\left(x^2+y^2\right)^{-3/2}$

(v) $\frac{-2y}{(x-y)^2}; \frac{2x}{(x-y)^2}$ (vi) $\frac{2x^3+3xy^2-y^3}{(x+y)^2}; \frac{2y^3+3xy^2-x^3}{(x+y)^2}$

(vii) ye^{xy}, xe^{xy} (viii) $\frac{1}{x}, \frac{1}{x}$]

2. For the following functions, find f_{xx}, f_{yx}, f_{xy}, f_{yy} :

(i) $x^2 + 2xy + y^2$, (ii) $x^3 + 3x^2y + 3xy^2 + y^3$, (iii) e^{x2+y2} (iv)x^2y^2.

[Ans. (i) 2 (every case) (ii) 6 (x + y) in every case

(iii) $2e^{x3+y2}(l + 2x^2)$;$4xye^{x2+y3}$;$4xye^{x2+y2}$: $2e^{x2+y2}$ $(1 + 2y^2)$ (iv) $2y^2$; 4xy ; 4xy ; $2x^2$]

3. If f(x, y) = e^{x+y}, then $\frac{\partial f}{\partial y} = f(x,y), \frac{\partial f}{\partial x} = f(x,y)$.

4. If f(x,y) = $\frac{x-y}{x+y}$, then $\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y}$, atx = y = l.

5. If f(x, y, z) = $x^2 + y^2 + z^2$, then $f_{xx} = f_{yy} = f_{zz}$.

6. If u = $\sqrt{x^2+y^2}$ Vx^2+y^2, findu_{xx},u_{yy} [Ans. $\frac{y^2}{(x^2+y^2)^{3/2}}; \frac{x^2}{(x^2+y^2)^{3/2}}$]

7. If = log ($x^2 + y^2$) show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

8. If (x, y) = $\frac{y}{x}$ log show that $f_{xy} = f_{yx}$

9. If u = x^2 + y'' + z , find the value of $xu_x + yu_y + zu_z$. [Ans. 2]

10. If u = $x^2y + y^2z + z^2x$, find the value of $u_x + u_y + u_z$. [Ans. $(x + y + z)^2$]

LAGRANGES METHOD MULTIPLIERS :

To find the extreme of a differentiable function f(x, y) of two variables subject to the condition of independent and differentiable equation g (x, y) = 0

Here f(x, y) depends in reality on only two independent variables x and y. For symmetry multiply f(x, y) by 1 and equ. (i) by l (a constant) and add them together so that we get,

L = 1. f (x, y) + l g (x, y)

For maximum or minimum $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0$ … (ii)

Now solving equ. (i) and (ii) we may find the values of z and y, and hence the extremes.

Example 100 : Find the extreme value of the function

$f(x, y) = x^2 - y^2 + xy + 5x$. Subject to x + y + 3 = 0

Let g (x, y) = x + y + 3

$L = 1. f(x, y) + l\, g(x, y) = x^2 - y^2 + xy + 5x + l(x + y + 3)$.

For max. or min., $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0$

i.e., 2x + y + 5 + l = 0 … (i), – 2y + x + l = 0 …(ii)

Again x + y + 3 = 0 (given) …(iii) Solving (i) (ii) (iii)

We get x = – 2, y = – 1, l = 0 so at (– 2, – 1)

The given function has extreme value and the value is

f (– 2, – 1) = 4 – 1 + 2 – 5.2 = 6 – 11 = – 5.

Example 101: Find the extreme values of the function.

$f(x, y) = x^2 - y^2 + xy + 5x$ subject to x – 2y = 0

Let L = 1 f(x, y) + lg (x, y), where g (x, y) = x – 2y; $L = x^2 - y^2 + xy + 5x + l(x - 2y)$.

From max. or min. $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0$ so that $\frac{\partial L}{\partial x} = 2x + y + 5 + l = 0$ … (i)

$$\frac{\partial L}{\partial x} = -2y + x - 2\lambda = 0 \quad \text{… (ii)}$$

In equ. (i), putting x = 2y as x – 2y = 0 we find 4y + y + 5 + l or, l = 0

From (iii), 5y + 5 = 0 or, y = – 1 as l = 0 ∴ x = 2y = 2 (–1) = – 2

At (– 2, – 1), extreme value = 4 – 1 + 2 + 5 (– 2) = – 5.

Example 102: Find the minimum value of $f(x, y) = x^2 + y^2$ subject to x + y = 10

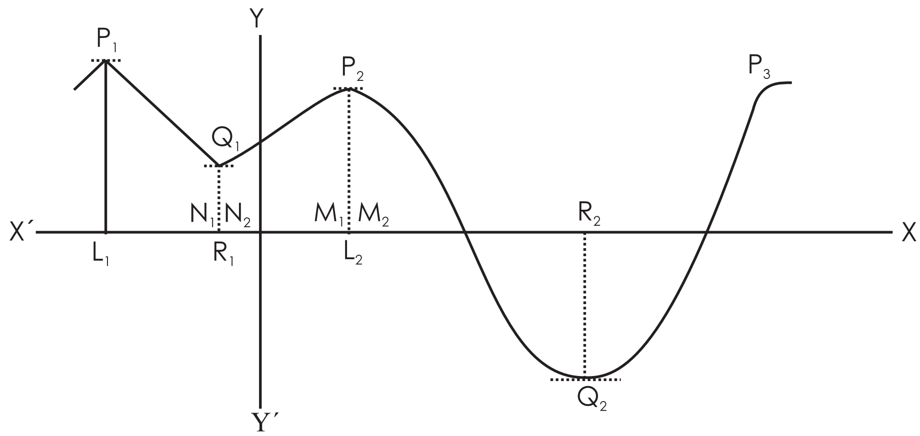
Let g (x) = x + y = 10. $L = 1. f(x) + lg(x) = x^2 + y^2 + l(x + y - 10)$.

For minimum value $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0$ i.e. 2x + l = 0 …. (i) 2y + l = 0 … (ii)

Again x + y – 10 = 0 …. (iii) solving (i), (ii), (iii) we get x = 5, y = 5, l = – 10. At x = 5, y = 5, the given function f(x, y) has minimum value and the value is $5^2 + 5^2 = 50$.

3.4.6 MAXIMUM AND MINIMUM

A function f(x) is said to be maximum at x = a if f(a) is greater than every other value of f(x) in the immediate neighbourhood of x = a (i.e., f(x) ceases to increase but begins to increase at x = a. Similarly the minimum value of f(x) will be that value at x = b which is less than other values in the immediate neighbourhood of x = b. [i.e., f(x) ceases to decrease but begins to increase at x = b]



The above figure represents graphically a continuous function f(x). The function has a maximum values at P_1, P_2, P_3 and also minimum values at Q_1, Q_2. For P_2, abscissa is OL_2, ordinate is $P_2 L_2$. Similarly OR_1 and R_1Q_1 are the respective abscissa and ordinate to Q_1. In the immediate neighbourhood of L_2, we may get a range of $M_1 L_2 M_2$ (on either side of L_2) such that for every value of x within that range (expect at L_2), the value f(x) is less than $P_2 L_2$ (i.e., the value at L_2), . Hence we can now show that f(x) is maximum at x = OL_2. In the same way we may find a neighbourhood $N_1 R_1 N_2$ or R_1 so that for every value of x within the range (expect at R_1) the value of f(x) is greater than at R_1. So the function is minimum at R_1.

The ordinate $P_2 L_2$ should not necessarily be bigger than the ordinate $R_1 R_1$.

Features regarding Maximum and Minimum : (i) Function may have several maximum and minimum values in an interval (as shown in Fig. Above).

(ii) Maximum and minimum values of a function occur alternatively (for clear idea see Fig above).

(iii) At some point the maximum value may be less than the minimum value (i.e., Fig., $P_2 L_2 < Q_2 R_2$).

(iv) In the graph of the function maxima are like mountain tops while minima are like valley bottoms.

(v) The points at which a function has maximum or minimum value are called turning points and the maximum and minimum values are known as extreme values, or extremum or turning values.

(vi) The values of x for which f(x) = 0 are often called critical values.

Criteria for Maximum and Minimum : (a) If a continuous function y = f(x) is maximum at a point x = a (say), then by definition, it is an increasing function for values of x just before x = a and a decreasing function for values of x just after x = a, i.e., its derivative $\frac{dy}{dx}$ is positive before x = a and negative after x = a. This means at x = a, $\frac{dy}{dx}$ changes sign from + ve to – ve.

Since $\frac{dy}{dx}$ is continuous function of x it can change sign only after passing through zero value.

Thus $\frac{dy}{dx} = 0$.

Hence for a function y = f(x) to attain maximum value at x = 1.

(i) $\frac{dy}{dx} = 0$, (ii) $\frac{dy}{dx}$ changes sign from + ve to – ve at x = a, i.e., is a decreasing function of x and so $\frac{d^2y}{dx^2} < 0$.

(b) If again a continuous function y = f(x) is minimum at x = a, then by definition it is decreasing just before x = a and then increasing just after x = a, i.e., its derivative $\frac{dy}{dx}$ is – ve just before x = a and + ve just after x = a. This means $\frac{dy}{dx}$ changes sign from – ve to + ve values. A continuous function $\frac{dy}{dx}$ can change sign only after passing through zero value, so $\frac{dy}{dx} = 0$.

Hence for a continuous function y = f(x) to attain a minimum value at x = a,

(i) $\frac{dy}{dx} = 0$, (ii) $\frac{dy}{dx}$ changes sign from – ve at + ve at x = a, i.e., $\frac{dy}{dx}$ is an increasing function of x hence $\frac{d^2y}{dx^2} > 0$.

Summary :

For a function y = f(x) to attain a maximum point at x = a,

(i) $\frac{dy}{dx} = 0$, (ii) $\frac{d^2y}{dx^2} < 0$, and

for a minimum point

(i) $\frac{dy}{dx} = 0$ (ii) $\frac{d^2y}{dx^2} > 0$.

Conditions for Maximum and Minimum : Necessary Condition. If a function f(x) is maximum or minimum at a point x = b and if f′(b) exists then f′ (b) = 0.

Sufficient Condition : If b is a point in an interval where f(x) is defined and if f ′ (b) = 0 and f″ (b) ≠ 0, then f(b) is maximum if f ″ (b) <0 and is minimum if f ″ (b) > 0. (The proof is not shown at present).

Definition :

If in a function y = f(x), for continuous increasing value of x, y increases upto a certain value and then decreases, then this value of y is said to be the *maximum value*. If again y decreases upto a certain value and then increases for continuous increasing value of x, then this value y is said to be minimum value. The points on the curve y = f(x). which separate the function from its increasing state to decreasing state or vice versa are known as turning points on the curve. From these turning points the curve may attain the extreme values. (i.e., maximum or minimum).

Analytical Expression : Let $\alpha - h, \alpha, \alpha + h$ be the three values of x (h is very small) : then the corresponding values of y will be $f(\alpha - h)$, $f(\alpha)$ and $f(\alpha + h)$. If however, $f(\alpha)$ be greater than $f(\alpha - h)$ and $f(\alpha + h)$, then f(x) is said to be maximum at $x = \alpha$. Again if $f(\alpha)$ be less than both $f(\alpha - h)$ and $f(\alpha + h)$, the f(x) is said to be minimum at $x = \alpha$.

Working Rule : by First Derivative Method :

Steps to find the maximum or minimum point of a curve y = f(x).

Find $f'(x)$ and equate it to zero. From the equation $f'(x) = 0$, find the value of x, say α and β.

Here the number of roots of $f'(x) = 0$ will be equal to the number of degree of $f'(x) = 0$.

Then find $f'(\alpha - h)$ and $f'(\alpha + h)$, then note the change of sign if any (here h is very small).

If the change is from positive to negative, f(x) will be maximum at $x = \alpha$. If again the change of sign is from negative to positive, f(x) will be maximum at $x = \alpha$.

Similar treatment for $x = \beta$.

Note : We have seen that $\frac{dy}{dx}$ changes sign (positive to negative or vice versa) is passing through the value zero. It may also happen that $\frac{dy}{dx}$ changes sign in passing through an infinite value (the detail is not shown as present).

Example 103: Examine for maximum and minimum for the function $f(x) = x^3 - 27x + 10$.

Solution:

Now $f'(x) = 3x^2 - 27$. For maximum and minimum $f'(x) = 0$ or $3x^2 - 27 = 0$

or, $x^2 = \frac{27}{3} = 9 \therefore x = \pm 3$.

Now let us enquire whether f(x) is maximum or minimum at these values of x.

For x = 3, let us assign to x, the values of 3 – h and 3 + h (h is very small) and put these values at f(x).

Now $f'(3 - h) = 3(3 - h)^2 - 27$ which is negative for h is very small.

And $f'(3 + h) = 3(3 + h)^2 - 27$ which is positive.

Thus $f'(x)$ i.e., $\frac{dy}{dx}$ changes sign from negative to positive as it passes throug x = 3. Therefore f(x) is minimum at x = 3. The minimum value is

$$f(x) = 3^3 - 27.3 + 10 = 27 - 81 + 10 = -44.$$

Similarly $f'(-3 - h) = 3(-3 - h)^2 - 27$, it is positive and $f'(-3 + h) = 3(-3 + h)^2 - 27$, it is negative, and consequently the change of sign of f(x) being positive to negative, f(x) is maximum at x = – 3.

The maximum value is $f(-3) = (-3)^3 - 27(-3) + 10 = -27 + 81 + 10 = 64$.

Working Rule :

By second Derivative Method :

For the function y = f(x), find $\frac{dy}{dx}$ and make it zero. From the equation $\frac{dy}{dx} = 0$, find the values of x say a and b.

Find $\frac{d^2y}{dx^2}$. Put x = a in $\frac{d^2y}{dx^2}$, if $\frac{d^2y}{dx^2}$ at x = a is – ve, the function is maximum at x = a and maximum value is f(a).

If again by putting x = a in $\frac{d^2y}{dx^2}$, the result is +ve, then the function is minimum and minimum value is f(α). Similarly for the value x = b.

Example 104 : Examine maximum value of the function

$y = x^3 - 27x + 10$ (the same example given above)

Solution:

$\frac{dy}{dx} = 3x^2 - 27$. Taking $\frac{dy}{dx} = 0$ i.e., $3x^2 - 27 = 0$, we get x = ± 3.

Now $\frac{d^2y}{dx^2} = 6x$. At x = 3, $\frac{d^2y}{dx^2} = 6.\ 3 = 18$, + ve, so the function is minimum at x = 3 and min. value is $3^3 - 27.3 + 10 = -44$.

Again at $x = -3, \frac{d^2y}{dx^2} = 6.(-3) = -18$, –ve, so the function is max. at x = – 3 and max. value is $(-3)^3 - 27(-3) + 10 = 64$.

Example 105 : For what value of x the following function is maximum or minimum, $\frac{x^2 - 7x + 6}{x - 10}$.

(Use of Second Derivative Method)

Let y = $\frac{x^2 - 7x + 6}{x - 10}$

$$\frac{dy}{dx} = \frac{(x-10)(2x-7) - (x^2 - 7x + 6)}{(x-10)^2} = \frac{x^2 - 20x + 64}{(x-10)^2}$$

making $\frac{dy}{dx} = 0$ we find $x^2 - 20x + 64 = 0$

or, $(x - 4)(x - 16) = 0 \therefore x = 4, 16$.

Now $$\frac{d^2y}{dx^2} = \frac{(x-10)^2(2x-20) - (x^2 - 20x + 64).2(x-10)}{(x-10)^3}$$

$$= \frac{2\{(x-10)^2 - (x-4)(x-16)\}}{(x-10)^3}$$

At x = 4, $\frac{d^2y}{dx^2} = \frac{2(4-10)^2}{(4-10)^3} = \frac{2}{-16} = -\frac{1}{3}$, – ve $\therefore$ function is max. at x = 4.

At x = 16, $\frac{d^2y}{dx^2} = \frac{2(16-10)^2}{(16-10)^3} = \frac{2}{6} = \frac{1}{3}$ + ve $\therefore$ function is min. at x = 16.

Application :

Few terms : The term **marginal cost** indicates the changes in the total cost for each additional unit of production.

If total cost = c, output = q, then c = f(q) and $\frac{dc}{dq} =$ marginal cost.

Now, average cost $= \frac{c}{q} = \frac{f(q)}{q}$

For example, let c = $q^3 - 2q^2 + 4q + 15$, to find average cost and marginal cost. Here c is a function of q.

Total cost c = $q^3 - 2q^2 + 4q + 15$

$$\text{Average cost} = \frac{c}{q} = \frac{q^3 - 2q^2 + 4q + 15}{q} = q^2 - 2q + 4 + \frac{15}{q}$$

$$\text{Marginal cost} = \frac{dc}{dq} = \frac{d}{dq}\left(q^3 - 2q^2 + 4q + 15\right) = 3q^2 - 4q + 4$$

Note : The total cost is represented by the constant 15, for even if the quantity produced is zero., the cost equal to 15 will have to be incurred by the firm.

Again since this constant 15 drops out during the process of deriving marginal cost, obviously the magnitude of fixed cost does not affect the marginal cost.

Minimum Average Cost : The minimum average cost can be determined by applying first and second derivatives, which will be clear from the following example.

Example 106 : If the total cost function is c = $3q^3 - 4q^2 + 2q$, find at what level of output, average cost be minimum and what level will it be?

Solution:

Total cost (= TC) = $3q^3 - 4q^2 + 2q$.

$$\text{Average cost } (= AC) = \frac{c}{q} = 3q^2 - 4q + 2,$$

$$\text{Marginal cost (MC)} = \frac{d(TC)}{dq} = 9q^2 - 8q + 2$$

Now $\frac{d(AC)}{dq} = 6q - 4$; making $\frac{d(AC)}{dq} = 0$ we get $6q - 4 = 0$, $q = \frac{2}{3}$.

At this level average cost function will be minimum if $\frac{d^2}{dq^2}(AC) > 0$. Now $\frac{d^2}{dq^2}(AC) = 6 > 0$ which shows average cost is minimum. Hence average cost will be minimum at an output level of $\frac{2}{3}$ and its value will be $3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 2 = \frac{2}{3}$.

Marginal Revenue : For any demand function p = f (q), the total revenue (TR) is the product of quantity demand (q), and the price (p) per unit of output.

TR = q x p = q x f (q) (as p = f(q))

Now the marginal revenue represents the change in TR for each additional unit of sale, so

Marginal revenue $(MR) = \frac{d(TR)}{dq}$ i.e., derivative of TR w.r.t. quantity demanded.

Here TR = Revenue = pq, AR (average revenue) $= \frac{pq}{q}$

For TR maximum we may also have to find the output (q), making MR = 0 i.e., $\frac{d(TR)}{dq} = 0$ (i.e., first derivative w.r.t. output equal to zero) and hence we can estimate the price (p) and finally the maximum revenue.

Note : For profit maximisation, MR = MC.

Example 107 : A radio manufacturer produces 'x' sets per week at a total cost of Rs. $\frac{x^2}{25} + 3x + 100$. He is monopolist and the demand for his market is x = 75 – 3p ; where p is the price in rupees per set. Show that the maximum net revenue is obtained when about 30 sets are produced per week. What is the monopoly price?

Solution:

Net revenue (NR) = Sale – total cost = x × p – TC

$$= x \quad \frac{75 - x}{3} \quad - \quad \frac{x^2}{25} + 3x + 100$$

For max. net revenue, we have $\frac{d(NR)}{dx} = 0$ or, $25 - \frac{2x}{3} - \quad \frac{2x}{25} + 3 \quad = 0$

or, $\quad -2x \quad \frac{1}{3} + \frac{1}{25} \quad = 3 - 25 = -22$

or, $\quad x \quad \frac{28}{75} \quad = 11$ $\qquad$ or, $\quad x = \frac{11 \times 75}{28} = 30$ (app.)

Now, $p = \frac{75 - x}{3} = \frac{75 - 30}{3} = \frac{45}{3} = 15$

$\therefore$ monopoly price = ₹ 15 per set.

Example 108 : A manufacturer can sell x items per month at a price p = 300 – 2x rupees. Produced items cost the manufacturer y rupees y = 2x + 1000. How much profit will yield maximum profits?

Solution:

Profit (P) = Sale – total cost = x × p – y

= x (300 – 2x) – (2x + 1000) = 298x – $2x^2$ – 1000

For maximum profits, $\frac{dP}{dx} = 0$ i.e., 298 – 4x = 0

or, x = 74.5 = 74 (as the number cannot be fraction and also x ≠ 75 as x ≯ 74.5)

Again $\frac{d^2P}{dx^2} = -4 < 0.$ Hence the profit will be maximum for 74 items.

Alternative way :

For profit maximisation we know MR = MC i.e $\frac{d(TR)}{dx} = \frac{d(TC)}{dx}$ ... (i)

TR = px = (300 – 2x) x = 300x – $2x^2$

$$\frac{d(TR)}{dx} = \frac{d}{dx}\left(300x - 2x^2\right) = 300 - 4x$$

Again TC = 2x + 1000, $\frac{d(TC)}{dx}$ = 2. Now by (i), we get

300 – 4x = 2 or, 4x = 298 or, x = 74.5 = 74.

(as the number cannot be fraction and also x ≠ 75 as x ≯ 74. 5)

Example 109 : The demand function for a particular commodity is $y = 15e^{-x/3}$ for $0 \le x \le 8$ where y is the price per unit and x is the number of units demanted. Determine the price and the quantity for which the revenue is maximum.

Revenue (R) = xy = x. $15e^{-x/3}$ = 15x. $e^{-x/3}$

For maximum $\frac{dR}{dx} = 15 \quad x.e^{-x/3} \quad \frac{-1}{3} \quad + e^{-x/3} \quad = -5xe^{-x/3} + 15e^{-x/3}$

For $\frac{dR}{dx} = 0,$ we get – $5xe^{-x/3}$ + $15e^{-x/3}$ = 0

or, $5e^{-x/3}$ (3 – x) = 0, either $e^{-x/3}$ = 0, or, 3 – x = 0

i.e., x = ∞ (absurd) or x = 3

Also $\frac{d^2R}{dx^2} = -5 \quad x.e^{-x/3}. \quad -\frac{1}{3} \quad + e^{-x/3} \quad + 15e^{-x/3}. \quad -\frac{1}{3}$

For x = 3, $\frac{d^2R}{dx^2} < 0,$ it is maximum.

Hence the maximum profit is obtained by putting x = 3 in the revenue equation $R = 15xe^{-x/3}$

$= 15.3.e^{-1} = = \frac{45}{2.72} = 16.54.$

Example 110 : For a certain establishment, the total revenue function R and the total cost function C are given by

$R = 83x - 4x^2 - 21$ and $c = x^3 - 12x^2 + 48x + 11$ where x = output.

Obtain the out put for which profit is maximum.

Solution:

Profit (p) = Revenue – Cost

$= 83x - 4x^2 - 21 - (x^3 - 12x^2 + 48x + 11) = -x^3 + 8x^2 + 35x - 32$

For max. $\frac{dp}{dx} = 0$. i.e., $\frac{d}{dx}\left(-x^3 + 8x^2 + 35x - 32\right) = 0$

or, $-3x^2 + 16x + 35 = 0$ or, $3x^2 - 16x - 35 = 0$

or, $(x - 7)(3x + 5) = 0$ or, $x = 7, -5/3$

Again $\frac{d^2p}{dx^2} = \frac{d}{dx}\left(-3x^2 + 16x + 35\right) = -6x + 16$

For x $= 7, \frac{d^2p}{dx^2} = -6.7 + 16 = -42 + 16 = -26 < 0,$ maximum

∴ the profit is maximum at x = 7.

Example 111 : Find two positive numbers whose product is 64 having minimum sum.

Let the two positive numbers be x and y. By question xy = 64 or $y = \frac{64}{x}$.

Let s be the sum of numbers of that s = x + y

or, $s = x + \frac{64}{x}$. Now diff. w.r.t.x, we get, $\frac{ds}{dx} = 1 - \frac{64}{x^2}$. For maximum $\frac{ds}{dx} = 0$

i.e., $1 - \frac{64}{x^2} = 0$ or, $x^2 = 64$ or x, = 8, –8

$\frac{d^2s}{dx^2} = 0 + \frac{2.64}{x^3}$. At $x = 8, \frac{d^2s}{dx^2} = \frac{2.64}{8^3} > 0$, minimum

∴ s is minimum for x = 8, the other number $y = \frac{64}{x} = \frac{64}{8} = 8$

∴ reqd. positive numbersare 8 and 8.

Example 112 : The sum of two numbers is 12. Find the maximum value of their product.

Solution:

Let the two numbers be x and y, so that x + y = 12.

Product (P) = xy = x (12 – x) = 12x – x^2

$\frac{dp}{dx} = 12 - 2x.$

For product max.

$\frac{dp}{dx} = 0$ i.e., 12 – 2x = 0 or, x = 6. Agian $\frac{d^2p}{dx^2} = -2 < 0, \text{max}.$

∴ reqd. product = x (12 – x) = 6 (12 – 6) = 6.6 = 36

Example 113 : A wire of length 16cm is to form a rectanglle. Find the dimensions of a rectangle so that it has maximum area.

Solution:

Let length be x, breadth be y so that

2x + 2y = 16 or, x + y = 8 or, y = 8 – x

Area (A) = length × breadth = xy = x (8 – x)

$\frac{dA}{dx} = 8 - 2x$. For max. area we have

$\frac{dA}{dx} = 0$, or 8 – 2x = 0 or, x = 4; $\frac{d^2y}{dx^2} = -2 < 0, \text{max}.$

For x = 4, y = 8 – 4 = 4.

So the area is maximum for length = breadth is 4 cm i.e., rectangle is a square.

SELF EXAMINATION QUESTION

1. Find the which values of x the following functions are maximum and minimum:

   (i) $x(12 - 2x)^2$ (ii) $x^3 - 3x^2 - 9x + 5$

   (iii) x3 – 6x2 + 9x – 8 (iv) $\frac{x^2 + x + 1}{x^2 - x + 1}$

   (v) $x^3 - 9x^2 + 24x - 12$. [Ans. (i) 2, 6; (ii) –1, 3; (iii) 1, 3; (iv) 1, –1 (v) 2, 4]

2. Find the maximum and minimum values of the above example.

   [Ans. (i) 128, 0 (ii) 10, –22, (iii) –4, –8, (iv) $3\frac{1}{3}$, (v) 8, 4]

3. Show that the maximum value of $x + \frac{1}{x}$ is less than its minimum value

4. Show that $f(x) = x^3 - 3x^2 + 6x + 3$ has neither maximum nor a minimum.

5. Show that the function $y = x^{-3} - 3x^2 + 5$ has a maximum value at $x = 0$ and a minimum value at $x = 2$.

6. Show that the fuction $x^3 - 6x^2 + 12x + 50$ is either a maximum nor a minimum at $x = 2$.

7. Find for what values of x, following expression is maximum and minimum respectively $2x^3 - 21x^2 + 36x - 20$. Find also the maximum and minimum values.

[Ans. minimum at $x = 6$; maximum at $x = 1$; minimum value = –128; maximum value = –3]

8. Show that the function $x(1-x^2)$ attains the maximum value at $x = \frac{1}{\sqrt{3}}$ and minimum value at $x = -\frac{1}{\sqrt{3}}$

9. show that $y = x^{-3} - 8$ has neither a maximum nor a minimum value. Has the curve a point of inflexion.

[Ans. Yes at 0, –8]

10. Show that the function $f(x) = x^2 + \frac{250}{x}$ minimum value at $x = 5$.

11. Show that the function $f(x) = x^{-3} - 6x^2 + 9x - 8$ has a maximum value at $x = 1$ and a minimum value at $x = 3$.

12. (i) A steel plant produces x tons of steel per week at s total cost of

₹ $\frac{1}{3}x^3 - 7x^2 + 11x + 50$.

Find the output level at which the marginal cost attains its minimum (using the concept of derivative as used in finding extreme values). [Ans. 7]

(ii) A firm produces x tons valuable metal per month ata a total cost c given by

$c =$ ₹ $\frac{1}{3}x^3 - 5x^2 + 75x + 10$.

Find at what level of output the marginal cost attains its minimum. [Ans. 5]

13. The total cost of output x given by $c = \frac{2}{3}x + \frac{35}{2}$

Find: (i) cost when output is 4 units. (ii) average cost of output of 10 units.

(iii) marginal cost when output is 3 units. [Ans. $20\frac{1}{6}, 2\frac{5}{12}; \frac{2}{3}$]

14. The demand function faced by a afirm is $p = 500 - 0.2x$ and its cost function is $c = 25x + 10000$ (p = price, x = output and c = cost). Find the output at which the profits of the firm are maximum.

Also find the price it will charge. [Ans. $1187\frac{1}{2}$; ₹ 262.50]

15. A firm produces x units of output per week ata a total cost of ₹ $\frac{1}{3}x^3 - x^2 + 5x + 3$. Find the output levels at which the marginal cost and the average cost attains their respective minima. [Ans. 1, 3/2]

16. A radio manufacture finds that he can sell x radios per week ar Rs. p each, where $p-2\ \ 100-\frac{x}{4}$.

His cost of productiono of x radios per week is Rs. $120x+\frac{x^2}{2}$.

show that his profit is maximum when the production is 40 radios per week. Find also his maximum profit per week. [Ans. ₹ 1600]

17. The total cost function of a firm is $c\frac{1}{x}x^3-3x^2+10x+10.$ Where c is the rotal cost and x is output.

A tax at the rate of Rs. 2 per unit of output is imposed and the producer adds it to his ost. If the market demand function is given by p = 2512 – 3x, where p is the price per unit of output, find the profit maximumsing output and hence the price. [Ans. 50 ; rs. 2362]

18. Find two positive numbers whose product is 16 having minimum sum [Ans. 4,4]

19. The sum of two numbers is 18. Find the maximum value of their product. [Ans. 81]

20. Find two positive numbers whose sum is 15 and the sum of whose square is minimum. [Ans. $\frac{15}{2},\frac{15}{2}$]

21. Of all the rectangles, each of which has perimeter 40cm., find the one having maximum area. [Ans. Square of side 10cm ; 100 sq. cm.]

22. A farmer can afford to buy 800 metres of wire fencing. He wishes to enclose a rectangular field of largest possible area. What should the dimensions of the field be? [Ans. 200m; 200m]

OBJECTIVE QUESTION

1. If $y = (2 - x)^2$ find dy/dx [Ans. x – 4]

2. If $y = x^3$ find $\frac{dy}{dx}^2+1$, when x = 1 [Ans. 10]

3. Differentiate x^6 w.r.t.x^2 [Ans. $3x^4$]

4. If y = log (4x) find dy/dx [Ans. $\frac{1}{x}$]

5. If $y=\left(\sqrt{x}+1\right)^2$ find dy/dx [Ans. $1+\frac{1}{\sqrt{x}}$]

6. If $y=\sqrt{x}+\frac{1}{\sqrt{x}}$ find find $2x\frac{dy}{dx}+y$ [Ans. $2\sqrt{x}$]

7. If y = logx find d^2y/dx^2 [Ans. $-\frac{1}{x^2}$]

8. If $y = 5^x$ find d^2y/dx^2 [Ans. $5^x (\log 5)^2$]

9. If $f(x) = 2x^3 + 3x^2 - 12x$ for what value of x will $f'(x) = 0$ [Ans. 1, –2]

10. If $y = x^3$, evaluate $1+\frac{d^2y}{dx^2}$ when x =–1 [Ans. –5]

11. Find $\frac{dy}{dx}$ for $2^x = t^2$ and $3y = t^3$ [Ans. t]

12. If x = at, y = $\frac{a}{t}$ find $\frac{d^2y}{dx^2}$ [Ans. $\frac{2}{at^3}$]

13. If $x = at^2$, y = 2at find d^2y/dx^2 [Ans. $\frac{-1}{2at^3}$]

14. If y = $x^{1/x}$ find dy/dx [Ans. $x^{1/x}.\frac{1-\log x}{x}$]

15. If y = $x^{\log x}$ find dy/dx [Ans. $x^{\log x}.\frac{2\log x}{x}$]

16. If e^{3x^2+5x-2} find dy/dx [Ans. $e^{3x^2+5x-2,}$ (6x +5)]

17. Given y = 2x2 – x + 1 find whether y is increasing, decreasing or stationary at x = $\frac{1}{4}$.

[Ans. Stationary at x = $\frac{1}{4}$]

18. For what value of x, $y = x^3 - 3x^2 - 9x + 5$ is minimum. [Ans. 3]

19. In the above example, for what value of x the functions is maximum? [Ans.–1]

20. If the total cost function is $c = q^2 - 2q + 5q$ find MC [Ans. $3q^2 - 4q + 5$]

21. The average cost function (AC) for certain commodity is AC = $-2x - 1 + \frac{50}{x}$ in terms of output x.

Find the MC [Ans. 4x –1]

22. In the above example, find the slope of MC [Ans. 4]

23. The average cost function (AC) for certain commodity is $AC = \frac{x}{2} - 4 + \frac{21}{x}$ find the slope of MC.

[Ans. x–4]

24. Examine $f(x) = x^3 - 6x^2 + 9x - 18$ for maximum or minimum values. [Ans. max. at x = 1, min. at x =– 3]

25. If $y = Ae^{kx} + be^{-kx}$ evaluate $y^2 - k^2y$ [Ans. 0]

26. If the cost functions is $c = \frac{q^3}{3} - 2q + 12$ find average variable cost. [Ans. $\frac{1}{3}q^2 - 2$]

27. The cost function (c) of a firm is as follows:

$c = \frac{2}{3}q^3 - \frac{3}{2}q^2 + 4q + 2.$ Is the slope of AC = $\frac{1}{q}$ (MC – AC). [Ans. Yes]

3.5 INTEGRATION

Definition:

For differential caculus we know. $\frac{d}{dx}\left(x^2\right) = 2x$ Here for a certain given function x^2, we have calculated its differential co-efficient w.r.t.x. The reverse process is known as integration, i.e. a certain function is given to us and we are required to find another function of the same variable whose differential co-eficient (w. r. t. the same variable) is the given function. For *example,* let the given function be 2x, we are to find another function (of the same variable x) whose differential co-efficient: (w. r. t. x) is 2x. Now the function will be x^2 and we shall say that the integration of 2x w. r. t. x is x^2. **Sign.** Let *f(x)* and ϕ (x) be two functions of x so that derivative of ϕ (x) w. r. t. x. *is f(x), i.e.,*

$\frac{d}{dx}\phi(x) = f(x)$ then the integral *of f(x)* is ϕ(x), which is expressed by attaching the sign of integration *before f(x)* and attaching *dx* after *f(x),* indicating that x is the variable of integration.

So if $\frac{d}{dx}\phi(x) = f(x)$ then $f(x)dx = \phi(x)$.

The function *f(x)* which is to be integrated is called the **integrand.**

Here $f(x)dx$ indicates in the *indefinite integral of (fx)* w. r. t. x.

Note. (i) We find integration is the inverse process of differention.

(ii) $\frac{d}{dx}$ ( ) and ( ) dx, the symbols are reverse to each other.

Constant of Integration

We know $\frac{d}{dx}x^2 = 2x; \frac{d}{dx}(x^2+5) = \frac{d}{dx}(x^2) + \frac{d}{dx}(5) = 2x + 0 = 2x \text{and} \frac{d}{dx}(x^2+c) = 2x$

In general, if $\frac{d}{dx}\phi(x) = f(x), \text{then} \frac{d}{dx}[\phi(x) + c] = f(x)$

f(x) dx = ϕ *(x)* + *c,* where c is a constant. (c is also known as constant of integration)

General Theorems concerning Integration : (A) The integral of the algebraic sum of a finite Number of functions is equal to the algebraic sum of their integrals. In

$(u_1 \pm u_2 \pm u_3 \ldots\ldots \pm u_n)\, dx = u_1\text{-}dx \pm u_2\text{-}dx \pm \ldots\ldots \pm u_n\, dx.$

Where u_1, u_2 , u_n are all functions of x or constants.

Example 114: $\int (x^4 \pm x)\, dx = \; x^4\, dx \pm xdx$

(B) A constant factor may be taken out from under the sign of integration and written before it. In symbols, $Audx = A\; udx$

Example 115: $\int (x^4 \pm x)\, dx = \int x^4 dx \pm \int xdx$

(C) $(A_1 u_1 \pm A_2 u_2 - \pm .... \pm A_n u_n)\, dx = A_1\; u_1\, dx \pm A\text{-}_2\; u_2\, dx \pm \; ..... \pm A_n\; le_n\, dx.$

Where A_1 A_2, A_n are constants and u1, u_2 ..u_n are all functions of x.

Example 116: $(x \pm 3x^2)\, dx = xdx \pm 3\; x^2.\, dx.$

Table of some fundamental integrals. The knowledge of differentiation will be employed now to find the indefinite integral of number of function. The constant integration will be understood in all cases.

Farmulae :

1. $x^n dx = \dfrac{x^{n+1}}{n+1} (n \neq -1)$ as $\dfrac{d}{dx}\; \dfrac{x^{n+1}}{n+1} = x^n$

2. $dx = x + c;$

3. $\dfrac{dx}{x^n} = -\dfrac{1}{(n-1)x^{n-1}}, (x \neq 1) + c$ **(corr. of formula 1)**

4. $\dfrac{dx}{x} = \log x + c$

5. $e^{mx}\, dx = \dfrac{e^{mx}}{m}\; (m \neq 0)$ corr, $e^x\, dx = e^x$

6. $a^x dx = \dfrac{a^x}{\log_e a} + c$ **(a > 0), a ±1)**

7. $a^{mx} dx = \dfrac{a^{mx}}{m \log_e a} + c,$ **(a > 0), a ±1)**

Standard Methods of Integration. The different methods of integration aim to reduce the given integral to one of the above fundamental of known integral, Mainly there are two principle processes: *(i) The method of substitution, i.e* , change of independent variable.

(ii) Integration by parts

If the integram is a rational fraction, it may be broken into partial fractions by algebra and then to apply the previous method for integration.

Example 117: Integrate the following w.r.t.x.

(i) x^4 (ii) x^{100} (iii) x (iv) 1 (v) –7 (vi) $x^{-4/5}$ (vii) $\sqrt[3]{x^4}$

Solution:

(i) $x^4 dx = \dfrac{x^{4+1}}{4+1} + c$ (by Formulae 1) $= \dfrac{1}{5}x^5 + c$

(ii) $x^{100}dx = \frac{x^{100+1}}{100+1} = \frac{x^{101}}{101} + c$ (iii) $xdx = \frac{x^{1+1}}{1+1} = \frac{x^2}{2} + c$

(iv) $1.dx = x + c$

(v) $x^{-7}dx = \frac{x^{-7+1}}{-7+1} = \frac{x^{-6}}{-6} = \frac{1}{-6x^6} + c$ (see formula 3)

(vi) $x^{-4/5}dx = \frac{x^{-4/5+1}}{-\frac{4}{5}+1} = \frac{x^{1/5}}{\frac{1}{5}} = 5x^{1/5} + c$

(vii) $\sqrt[3]{x^4}dx = x^{4/3}dx = \frac{3}{7}x^{7/3} + c$

Note. (i) The arbitrary constant of integration (c) may be given in the final step.

(ii) Check: $\frac{d}{dx}\ \frac{1}{5}x^5 + c\ = \frac{1}{5}\frac{d}{dx}x^5 + \frac{d}{dx}c = \frac{1}{5}.5x^4 + 0 = x^4$ (See Ex. 1)

(iii) So we find $x^4dx = \frac{1}{5}x^5 + c$, and $\frac{d}{dx}\ \frac{1}{5}x^5 + c\ = x^4$

Example 118 : (Integrate)

(i) $2x^4dx$ (ii) $(2x^{-3} + x^2)dx$

(iii) $3x^3 + \sqrt{x} - \frac{1}{x}\ dx$ (iv) $(a + x)^2dx$

Solution:

(i) $2x^4dx = 2\ x^4dx$ (by B) $= 2.\frac{x^{4+1}}{4+1} = \frac{2}{5}x^5 + c$

(ii) $(2x^{-3} + x^2)dx = 2x^{-3}dx + x^2dx$ (by A)

$= 2\ x^{-3}dx + x^2dx$ (by B) $= 2.\frac{x^{-3+1}}{-3+1} + \frac{x^{2+1}}{2+1} = -x^{-2} + \frac{1}{3}x^3 + c$

(iii) $3x^3 + \sqrt{x} - \frac{1}{x}\ dx = 3\ x^3dx + x^{1/2}dx - \frac{dx}{x}$ (by A and B)

$= \frac{3x^4}{4} + \frac{x^{1/2+1}}{\frac{1}{2}+1} - \log x$

$= \frac{3}{4}x^4 + \frac{2}{3}x^{3/2} - \log x + c.$

(iv) $(a+x)^2\ dx = \ (a^2+2ax+x^2)\ dx = a^2\ \ dx + 2a\ \ xdx + \ \ x^2dx$

$$= a^2x + 2a\frac{x^2}{2} + \frac{x^3}{3} = a^2x + ax^2 + \frac{1}{3}x^2 + c.$$

Example 119: Find the value of:

(i) $x\sqrt{x} - 4\sqrt{x} + \frac{5}{\sqrt{x}}\ \ dx$ (ii) $x^2\left(\sqrt{x} - x + 1\right)dx$

Solution:

(i) $(x\sqrt{x} - 4\sqrt{x} + \frac{5}{\sqrt{x}})\ dx = \ \ x\sqrt{xdx} - 4\ \ \sqrt{xdx} + 5\ \ \frac{dx}{\sqrt{x}}$

$$= \ \ x^{3/2}dx - 4\ \ x^{1/2}dx + 5\ \ x^{-1/2}dx$$

$$= \frac{x^{3/2+1}}{\frac{3}{2}+1} - 4.\frac{x^{1/2+1}}{\frac{1}{2}+1} + 5.\frac{x^{-1/2+1}}{-\frac{1}{2}+1}$$

$$= \frac{2}{5}x^{5/2} - 4.\frac{2}{3}x^{3/2} + 5.2x^{1/2} = \frac{2}{5}x^{5/2} - \frac{8}{3}x^{3/2} + 10\sqrt{x} + c$$

(ii) $x^2\left(\sqrt{x} - x + 1\right)\ dx = \ \left(x^{5/2} - x^3 + x^2\right)dx = \ \ x^{5/2} - \ \ x^3dx + \ \ x^2dx$

$$= \frac{x^{5/2+1}}{\frac{5}{2}+1} - \frac{x^4}{4} + \frac{x^3}{3} = \frac{2}{7}x^{7/2} - \frac{x^4}{4} + \frac{x^3}{3} + c.$$

Problem of algebraic functions:

To integrate a fraction algebraic expression of which the numerator is a polynomial function and the denominator is a monomial (or binomial) function, simplify the expression first to partial fraction.

Example 120: (integrate)

(i) $\frac{2x^2 + 3x^3 + 4}{x}dx$ (ii) $\frac{(x+2)^2}{\sqrt{x}}dx$ (iii) $\frac{2x^2 - 14x + 24}{x-3}dx$

Solution:

(i) Same as $\frac{2x^2 + 3x^3 + 4}{x}dx = 2\ \ \frac{x^2}{x}dx + 3\ \ \frac{x^3}{x}dx + 4\ \ \frac{dx}{x} = 2\ \ xdx + 3\ \ x^2dx + 4\ \ \frac{dx}{x}$

$$= 2\frac{x^2}{2} + 3.\frac{x^3}{3} + 4\log x = x^2 + x^3 + 4\log x + c.$$

(ii) $\frac{(x+2)^2}{\sqrt{x}}dx = \frac{x^2+4x+4}{x^{1/2}}dx = \frac{x^2}{x^{1/2}}dx + 4\ \frac{x}{x^{1/2}}dx + 4\ \frac{dx}{x^{1/2}}$

$= \frac{2}{5}x^{5/2} + 4.\frac{2}{3}x^{3/2} + 8x^{1/2} = \frac{2}{5}x^{5/2} + \frac{8}{3}x^{3/2} + 8x^{1/2} + c$

(iii) $\frac{2x^2-14x+24}{x-3}dx = \frac{(x-3)(2x-8)}{(x-3)}dx = (2x-8)dx = 2\ xdx - 8\ dx = 2.\frac{x^2}{2} - 8x$

$= x^2 - 8x + c$

Example 121: Evaluate $\frac{(4x-3)^3}{x^2}$

$\frac{(4x-3)^3}{x^2} = \frac{64x^3-144x^2+108x-27}{x^2}dx = \quad 64x - 144 + 108\frac{1}{x} - 27\frac{1}{x^2}\ dx$

$= 64\frac{x^2}{2} - 144x + 108\log x - 27.\frac{1}{x} = 32x^2 - 144x + 108\log x + 27\frac{1}{x} + c$

Example 122: Finid $e^{4x}dx$

$e^{4x}dx = \frac{e^{4x}}{4} + c.$ (by formula 5)

Example 123: Evaluate $e^{2\log x}dx$

$e^{2\log x}dx. = \ e^{\log x^2}dx = \ x^2dx = \frac{x^3}{3} + c.$

Example 124: Evaluate $\frac{e^{5x}+e^{3x}}{e^{4x}}dx$

Expression = $\frac{e^{5x}}{e^{4x}} + \frac{e^{3x}}{e^{4x}}\ dx = \ (e^x + e^{-x})dx$

$e^xdx + \ e^{-x}dx = e^x + \frac{e^{-x}}{-1} = e^x - e^{-x} + c.$

Example 125: Find the value of $\frac{e^{3x}+e^{5x}}{e^x+e^{-x}}dx$

Expression = $\frac{e^{4x}(e^{-x}+e^x)}{(e^x+e^{-x})}dx = \frac{e^{4x}(e^x+e^{-x})}{(e^x+e^{-x})}dx = \ e^{4x}dx = \frac{e^{4x}}{4} + c.$

Example 126: If $\frac{dy}{dx} = x^3 - 3x^2 + 1$ and y = 2 when x = 1, find y.

or, $y = x^3dx - 3\ x^2dx + \ dx = \frac{x^4}{4} - 3\frac{x^3}{3} + x + c$

or, $2 = \frac{1}{4} - 1 + 1 + c$ or, $2 = \frac{1}{4} + c$ or, $c = 2 - \frac{1}{4} = 7/4$

$\therefore \quad y = \frac{x^4}{4} - x^3 + x + \frac{7}{4}$

SELF EXMINATION QUESTIONS

1. (i) x^{11} (ii) x^{410} (iii) x (iv) x^a (v) a (vi) 2.

Ans. (i) $\frac{x^{12}}{12}$ (ii) $\frac{x^{411}}{411}$ (iii) $\frac{x^2}{x}$ (iv) $\frac{x^a + 1}{a+1}$ (v) ax (vi) 2x;+c in all the case

2. (i) x^{-11} (ii) x^{-n} (iii) $\frac{1}{x^n}$ (iv) $\frac{1}{x^2}$ (v) $\frac{1}{x}$ (vi) x^{-1}

Ans. (i) $\frac{x^{-10}}{-10}$ (ii) $\frac{x^{-n+1}}{-n+1}$ (iii) $\frac{x^{-n+1}}{-n+1}$ (iv) $\frac{-1}{x}$ (v) and (vi) logx

3. (i) $x^{3/2}$ (ii) $\sqrt{x^5}$

Ans. (i) $\frac{2}{5}x^{5/2}$ (ii) $x^3 + \frac{4}{5}x^{5/2} + \frac{x^2}{2}$

Integrate :

4. (i) $x\sqrt{x}dx$ (ii) $\left(x\sqrt{x}\right)^2 dx$

Ans. (i) $\frac{2}{5}x^{5/2}$ (ii) $x^3 + \frac{4}{5}x^{5/2} + \frac{x^2}{2}$

5. (i) $\frac{x^4 + 2x^2 + 1}{x^3}dx$ (ii) $\frac{(x-3)^2}{\sqrt{x}}dx$ (iii) $\frac{2\left(\sqrt{x}+1\right)^2}{x\sqrt{x}}dx$

Ans. (i) $\frac{x^2}{2} + 2\log x - \frac{1}{2x^2}$ (ii) $\frac{2}{5}x^{5/2} - 4x^{3/2} + 18\sqrt{x}$ (iii) $4\sqrt{x} + 4\log x - \frac{4}{\sqrt{x}}$

6. (i) $\frac{x^2 - 1}{x - 1}dx$ (ii) $\frac{x^3 - 1}{x - 1}dx$ (iii) $\frac{x^2 + 3x + 2}{x + 1}dx$ (iv) $\frac{x^2 - x - 6}{x + 2}dx$.

Ans. (i) $\frac{x^2}{2} + x$ (ii) $\frac{x^3}{3} + \frac{x^2}{2}$ (iii) $\frac{x^2}{2} + 2x$ (iv) $\frac{x^2}{2} - 3x$

7. (i) $x^{2x}dx$ (ii) $e^{-4x}dx$ (iii) $e^{-mx}dx$ (iv) $e^{2\log x}$ (v) $e^{m\log x}\,dx$

Ans. (i) $\frac{e^{2x}}{2}$ (ii) $\frac{e^{-4x}}{-4}$ (iii) $\frac{e^{mx}}{m}$ (iv) $\frac{x^3}{3}$ (v) $\frac{x^{m+1}}{m+1}$

8. (i) $5^x\,dx$ (ii) $5^{2x}dx$

Ans. (i) $\frac{5^x}{\log_e 5}$ (ii) $\frac{5^{2x}}{2\log_e 5}$

9. (i) $\frac{e^{4x}+e^{2x}}{e^{3x}}dx$ (ii) $\frac{e^{2x}+e^x+1}{e^x}dx$ (iii) $\frac{e^{3x}+e^x}{e^x+e^{-x}}dx$

(iv) $\frac{3e^{2x}+3e^{4x}}{e^x+e}dx$ (v) $\frac{1+e^{2x}}{e^x+e^{-x}}dx$

Ans. (i) $e^x - e^{-x}$ (ii) $e^x + x - e^{-x}$ (iii) $\frac{e^{3x}}{2}$ (iv) e^{3x} (v) e^x

Evaluate:

10. (i) $3\sqrt{x}+5+\frac{2}{x}\,dx$

Ans. $2x^{3/2}+5x+2\log x+c$

(ii) $\frac{(1-x)^3}{x}dx$

Ans. $\log x - 3x + \frac{3}{2}x^2 - \frac{x^3}{3} + c$

11. If $\frac{dy}{dx} = x^3 + x \quad + x + 1$ and if y = 2 when x = 1, find the value only.

Ans. $\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x-\frac{1}{12}$

12. If $f'(x)\,\frac{e^{3x}+e^{5x}}{e^x+e^{-x}}$ and $f(0) = \frac{1}{4}$, find *f(x)*.

Ans. $\frac{e^{4x}}{4}$

Hints. $f'(x) = \frac{e^{4x}(e^x+e^{-x})}{e^x+e^{-x}} = e^{4x}, f'(0) = \frac{1}{4} = \frac{1}{4} + c, c = 0$ & etc.

13. $\frac{e^{x-1}+e^{e-1}}{e^x+x^e}dx$

Ans. $\frac{1}{e}\log(e^x + x^e$

Hint. $I = \frac{1}{e}\;\frac{e.e^{x-1}+ex^{e-1}}{e^x+e^x}dx = \frac{1}{e}\;\frac{e^x+ex^{e-1}}{e^x+e^e}dx = \frac{1}{e}\;\frac{du}{u}, u = e^x + x^e$ & etc.

3.5.1 METHOD OF SUBSTITUTION

Let $I = \quad (x)dx$ then $\frac{dI}{dx} = f(x)$

Agian let $x = \phi(z)$, Then $\frac{dx}{dz} = \phi'(z)$., (change of variable)

Now, $\frac{dI}{dz} = \frac{dI}{dx}.\frac{dx}{dz} = f(x).\phi'(z) = f[\phi(z)].\phi'(z)$

$\therefore$ by def. $I = \int f[\phi(z)]\phi'(z)\,dz$ if $x = \phi(z)$.

The idea will be clear from the following example :

Integrate. $\int(2+3x)^n\,dx$. Let $2 + 3x = z \therefore 3\,dx = dz$

or, $\frac{dx}{dz} = \frac{1}{3}$ (i.e. here $\phi'(z) = \frac{dx}{dz} = \frac{1}{3}$)

Now, $I = \int z^n.\frac{1}{3}dz$, as $f[\phi(z)] = z^n$.

$\frac{1}{3}\int z^n\,dz = \frac{1}{3}.\frac{z^{n+1}}{n+1} = \frac{1}{3(n+1)}(2+3x)^{n+1} + c$. (putting $z = 2 + 3x$)

Note. It may be noted that there is no fixed rule for substitution in solving these types of problems.

Important Rules

1. $\int\frac{f'(x)}{f(x)}dx = \log_e f(x)$. **2.** $\int\{f(x)\}^p.f'(x)dx = \frac{1}{p+1}\{f(x)\}^{p+1}, (x \neq -1)$.

3. $\int e^x\{f(x) + f'(x)\}\,dx = e^x f(x)$.

Example 127: $\int\frac{dx}{2+x} = \log(2+x)$. Let $2+x = z$, $dx = dz$.

$I = \int\frac{dz}{z} = \log z = \text{Log}(2+x)$ as $\int\frac{dx}{x} = \log x$.

Example 128: $\int(2+x^2)^3.2x\,dx$

Let $2 + x^2 = z$

$2x\,dx = dz$

$I = \int z^3\,dz = \frac{1}{4}z^4$

$= \frac{1}{4}(2+x^2)^4$

Example 129: $\int e^x(x+1)dx$

Let, $e^x.x = z$

$(e^x + e^x.x)\,dx = dz$

or, $e^x(x+1)\,dx = dz$

$I = \int dz = z = e^x \cdot x$

Example 130:

$$(2x+5)^3\,dx$$

Let, 2x + 5 = u

2 . dx = du

$$dx = \frac{1}{2}\,du$$

$$I = \frac{1}{2}\;u^3\,du$$

$$= \frac{1}{2}\;\frac{1}{4}\cdot u^4$$

$$= \frac{1}{8}(2x+5)^4$$

Example 131:

$$\frac{dx}{5x}$$

Let, 5x = u

or, 5 dx = du

$$\text{or, } dx = \frac{1}{5}\,du$$

$$I = \frac{1}{5}\;\frac{du}{u} = \frac{1}{5}\log u$$

$$= \frac{1}{5}\log 5x + c$$

Example 132:

$$\frac{dx}{3-x}$$

Let, 3 – x = u

– dx = du

dx = – du

$$I = -\;\frac{du}{u} = -\log u$$

$$= -\log(3-x) + c$$

Example 133:

$$\frac{dx}{2+5x}$$

Let 2 + 5x = u

5.dx = du

$$dx = \frac{1}{5} du$$

$$I = \frac{1}{5} \frac{du}{u}$$

$$= \frac{1}{5} \log u$$

$$= \frac{1}{5} \log(2+5x)+c$$

Example 134: $\frac{2ax+b}{ax^2+bx+c}$ dx = $\frac{dz}{z} = \log z + c_1 = \log(ax^2+bx+c)+c_1.$

Let $ax^2 + bx + c = z$, $(2ax + b)$ dx = dz.

Note: Numerator is derivative of denominator.

Now $\frac{4x+b}{2x^2+3x+4}dx = \log(2x^2+3x+4)+c.$

$$\frac{3x^2+4x+1}{x^3+2x^2+x-1}dx = \log(x^3+3x^2+x-1)+c.$$

Example 135: $x^2\sqrt{1+x^3}dx$

Let $1 + x^3 = u^2$ (here u^2 is taken to avoid redical sign of square root).

Or, $3x^2dx = 2udu$ or, $x^2dx = \frac{2}{3} udu$

$$I = \frac{2}{3} u.udu \quad \left(as\sqrt{1+x^3} = \sqrt{u^2} = u\right)$$

$$= \frac{2}{3} u^2du = \frac{2}{3}.\frac{u^3}{3} = \frac{2}{9}\left(\sqrt{1+x^3}\right)^3 = \frac{2}{9}\left(1+x^3\right)^{3/2} + c.$$

Example 136: $\frac{xdx}{\sqrt{3x^2+1}}$. Let $3x^2 +1 = u^2$; $6xdx = 2udu$, $xdx = \frac{1}{3}udu$.

$$I = \frac{1}{3} \frac{udu}{u} = \frac{1}{3} du = \frac{1}{3}u = \frac{1}{3}\sqrt{3x^2+1}+c.$$

Example 137: $\frac{e^x - e^{-x}}{e^x + e^{-x}}dx$ Let $e^x + e^{-x} = u$, $(e^x - e^{-x})dx = du$

$I = \frac{du}{u} = \log u = \log(e^x + e^{-x}) + c$

Example 138: $\left(1 - \frac{1}{x^2}\right) e^{x+1/x}dx$ Let $x + \frac{1}{x} = u$, $\left(1 - \frac{1}{x^2}\right) dx = du$

$I = e^u du = e^u = e^{x+1/x} + c$

Example 139: $e^x\sqrt{e^x + 1}dx$

Let $e^x + 1 = u^2$, $e^x dx = 2u du$

$I = 2 \; u^2 du = \frac{2}{3}u^3 = \frac{2}{3}(e^x + 1)^{3/2} + c.$

Example 140: $\frac{dx}{x(\log x)^2}$ Let $\log x = z$, $\frac{dx}{x} = dz$

Now $I = \frac{dz}{z^2} = -\frac{1}{z} = \frac{-1}{\log x} + c.$

Example 141: $\frac{4x + 7}{2x + 3}dx$

$I = \frac{2(2x+3)+1}{2x+3}dx = 2 \; \frac{2x+3}{2x+3}dx + \frac{dx}{2x+3}$

$= 2 \; dx + \frac{dx}{2x+3} = I_1 + I_2$ where $I_1 = 2 \quad dx = 2x$

and $I_2 = \frac{1}{2} \; \frac{du}{u} = \frac{1}{2}\log u.$ Let $2x + 3 = u$, $2dx = du$

$= \frac{1}{2}\log(2x + 3)$

$I = 2x + \frac{1}{2}\log(2x + 3) + c.$

Example 142: $\frac{e^x}{x}(1+x\log x)\,dx, x > 0.$

$I = \; e^x \;\; \log x + \frac{1}{x} \;\; dx$

Let $e^x \log x = u$

$= \;\; du \; = u$

$e^x.\frac{1}{x} + e^x.\log x \;\; dx = du$

$= e^x \log x + c.$

or. $e^x \;\; \frac{1}{x} + \log x \;\; dx = du$

Example 143: $\frac{3x}{\sqrt{2x-1}}dx$ Let $2x - 1 = u^2$, $2\,dx = 2\,udu$, $dx = udu$

Again $2x = 1 + u^2$, $x = \frac{1}{2}\left(1+u^2\right)$, $3x = \frac{3}{2}\left(1+u^2\right)$

$I = \frac{3}{2} \;\; \frac{(1+u^2)u}{u}du = \frac{3}{2} \;\; du + \frac{3}{2} \;\; u^2 du = \frac{3}{2}u + \frac{3}{2}.\frac{u^3}{3}$

$= \frac{3u}{2} + \frac{u^3}{2} = \frac{3}{2}\sqrt{2x-1} + \frac{1}{2}(2x-1)^{3/2} + c.$

Example 144: Evaluate : $3x^2\sqrt{6x^3+11}dx$

Let $6x^3 + 11 = u^2$ so that $18x^2dx = 2udu$

$I = 3 \;\; \frac{2}{18}u.udu = \frac{1}{3} \;\; u^2du = \frac{1}{3}\;\frac{u^3}{3} \;\; = \frac{1}{9}\left(6x^3+11\right)^{3/2} + c.$

Example 145: Evaluate : $\frac{x+2}{\sqrt{x-2}}dx$ Let $x - 2 = u^2$, $dx = 2\,udu$. Again $x + 2 = 4 + u^2$.

$I = \;\; \frac{4+u^2}{u}.2udu = 2 \;\; \left(4+u^2\right)du$

$= 8 \;\; du + 2 \;\; u^2du = 8u + \frac{2}{3}u^3 = 8\sqrt{x-2} + \frac{2}{3}(x-2)^{3/2} + c.$

Example 146: Evaluate: $\frac{e^{-x}dx}{1+e^{-x}}$

Let $1 + e^{-x} = u$, $-e^{-x}dx = du$

$I = \;\; \frac{e^{-x}dx}{1+e^{-x}} = - \;\; \frac{du}{u} = -\log u = -\log(1+e^{-x})$

$$= -\log\ 1 + \frac{1}{e^x}\ \ = -\log\ \frac{e^x + 1}{e^x}\ \ = \log e^x - \log\left(e^x + 1\right) + c.$$

Example 147: Evaluate : $e^x\left(e^x + 1\right)^{1/2} dx$

Let $(e^x + 1)^{1/2} = u$, $e^x + 1 = u^2$, $e^x dx = 2udu$

$$I = 2\ u^2 du = \frac{2}{3}u^3 = \frac{2}{3}\left(e^x + 1\right)^{3/2} + c$$

Example 148: Evaluate : $\frac{x.e^{2x}}{(2x+1)^2} dx$

Let $\frac{e^{2x}}{(2x+1)} = u$ so that on differentiation

$$\frac{2(2x+1)e^{2x} - e^{2x} \cdot 2}{(2x+1)^2} dx = du, \frac{4xe^{2x}}{(2x+1)^2} dx = du$$

Now $I = \frac{1}{4}\ du = \frac{1}{4}u = \frac{1}{4}\frac{e^{2x}}{(2x+1)} + c$

Example 149: Evaluate : $\frac{2x-3}{\sqrt{4x-1}} dx$

let $4x - 1 = u^2$, $4dx = 2\ udu$, again $2x = \frac{1}{2}\left(1 + u^2\right), 2x - 3 = \frac{1}{2}\left(u^2 - 5\right)$

$$I = \frac{1}{2}\ \frac{u^2 - 5}{2u}.udu = \frac{1}{4}\ \left(u^2 - 5\right)du = \frac{1}{4}\ \frac{u^3}{3} - 5u$$

$$= \frac{1}{4}\ \frac{(4x-1)^{3/2}}{3} - 5\sqrt{4x-1}\ \ = \frac{1}{12}(4x-1)^{3/2} - \frac{5}{4}\sqrt{4x-1} + c$$

Example 150: $x^2\sqrt{3x^3 - 4}\ dx$

Let $3x^3 - 4 = u^2$, $9x^2dx = 2\ udu$

$$I = \frac{2}{9}\ u^2 du = \frac{2}{9}.\frac{u^3}{3} = \frac{2}{27}\left(3x^3 - 4\right)^{3/2} + c$$

3.5.2 INTEGRATION BY RATIONALISATION

In some cases rationalisation is required to avoid the surd in the numerator or denominator before integration. The idea will be clear from the following example.

Example 151: Integrate: $\dfrac{dx}{\sqrt{x+1}-\sqrt{x-1}}$

$$\text{Now } \frac{1}{\sqrt{x+1}-\sqrt{x-1}} = \frac{1}{\sqrt{x+1}-\sqrt{x-1}} \times \frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}$$

$$= \frac{\sqrt{x+1}+\sqrt{x-1}}{(x+1)-(x-1)} = \frac{1}{2}\left(\sqrt{x+1}+\sqrt{x-1}\right)$$

$$I = \frac{1}{2}(\sqrt{x+1}+\sqrt{x-1})\,dx = \frac{1}{2}\ \sqrt{x+1}\,dx + \frac{1}{2}\ \sqrt{x-1}dx$$

$$= \frac{1}{2}.\frac{(x+1)^{1/2+1}}{\frac{1}{2}+1} + \frac{1}{2}\cdot\frac{(x-1)^{1/2+1}}{\frac{1}{2}+1} = \frac{1}{3}(x+1)^{3/2} + \frac{1}{3}(x-1)^{3/2} + c.$$

<u>**SELF EXAMINATION QUSTIONS**</u>

1. $(2X+3)^4 D\chi$ — $\frac{1}{10}(2X+3)^5$, c is to be added in all answers

2. $(2-3X)^6 D\chi$ — Ans. $\frac{-1}{21}(2x-3)^7$

3. $\sqrt{2x+5}dx$ — Ans. $\frac{1}{3}(2x+5)^{3/2}$

4. $\sqrt[3]{3x+4}dx$ — Ans. $\frac{1}{4}(3x+5)^{4/3}$

5. $(3x+4)^{5/3}$ — Ans. $\frac{1}{8}(3x+4)^{8/3}$

6. $x\sqrt{x^2+1}dx$ — Ans. $\frac{1}{3}(x^2+1)^{3/2}$

7. $(2x+5)\sqrt{x^2+5x}dx$ — Ans. $\frac{2}{3}(x^2+5x)^{3/2}$

8. $\dfrac{xdx}{\sqrt{x^2-a^2}}$ — Ans. $\sqrt{x^2-a^2}$

9. $\dfrac{x^3dx}{\sqrt{x^2+1}}$ — Ans. $\frac{1}{3}(x^2+1)^{3/2} - \sqrt{x^2+1}$

10. $\dfrac{6xdx}{(4-x^2)^2}$ — Ans. $\dfrac{1}{4-3x^2}$

11. $(3x^2-5x+7)^m(6x-5)dx$ — Ans. $\dfrac{(3x^2-5x+7)^{m+1}}{3}$

12. $x\left(3x^2+7\right)^7 dx$ Ans. $\frac{1}{48}\left(3x^2+7\right)^8$

13. $\frac{xdx}{\sqrt{3x^2+4}}$ Ans. $\frac{\sqrt{3x^2+4}}{3}$

14. (i) $(2x+3)\sqrt{(x^2+3x-1)}dx$ Ans. $\frac{2}{3}\left(x^2+3x-1\right)^{3/2}$

(ii) $\frac{x-2}{\sqrt{2x-8x+5}}dx$ Ans. $\frac{1}{2}\sqrt{2x^2-8+5}$

15. (i) $\frac{t^2dt}{\sqrt[3]{t^4+3}}$ Ans. $\frac{3}{8}\left(t^4+3\right)^{2/3}$

(ii) $\frac{x-2}{\sqrt[3]{x^2-4x+5}}dx$ Ans. $\frac{3}{4}\left(x^2-4x+5\right)^{2/3}$

16. $\frac{dx}{x\log x}$ [Ans. logx]

17. $\frac{\log\sqrt{x}}{3x}$ Ans. $\left(\log\sqrt{x}\right)^2$

18. (I) $\frac{(x+1)(x+\log x)^2 dx}{x}$ Ans. $\frac{1}{2}(x+\log)^3$

(ii) $\frac{dx}{x\left\{12+7\log x+(\log)^2\right\}}$ [Ans. log (log x + 3) – log(log x + 4)]

(iii) $\frac{dx}{x(\log x)^2+4x\log x-12x}$ Ans. $\frac{1}{8}\log(\log-2)-\frac{1}{8}\log(\log x+6)$

19. (i) $\frac{2x-1}{3x+7}dx$ Ans. $\frac{2}{2}x-\frac{17}{9}\log(3x+7)$

(ii) $\frac{2x+3}{3x+2}dx$ Ans. $\frac{2}{3}x+\frac{5}{9}\log(3x+2)$

20. $\frac{a+bx}{c+dx}dx$ Ans. $\frac{bx}{d}+\frac{(ad-bc)}{a^2}\log(c+dx)$

21. $e^x\ \frac{1}{x}-\frac{1}{x^2}\ dx$ Ans. $\frac{e^x}{x}$

22. $e^x(x-2)\frac{dx}{x^3}$ Ans. $\frac{e^x}{x^2}$

23. (i) $\frac{e^x+e^{-x}}{e^x-e^{-x}}dx$ [Ans. $\log(e^x-e^{-x})$]

(ii) $\frac{1}{1+e^{x/2}}dx$ [Ans. $-2(e^{-x/2}+1)$]

[Hint. I = $\int \frac{e^{-x/2}}{e^{-x/2}+1}dx$, put $e^{-x/2}$ + 1 = u & etc.]

(iii) $\int \frac{e^{x-1}+x^{e-1}}{e^x+x^e}dx$ Ans. $\frac{1}{e}\log(e^x+x^e)$

(iv) $\int \frac{e^{2x}}{2+e^x}dx.$ [Ans. x^x – 2 log (2 + e^x)]

Hint. $I=\int \frac{e^x dx}{2+x}=\int \frac{u du}{2+u}, e^x = u$ & etc.

24. (i) $\int \frac{x dx}{\sqrt{2x+3}}$ Ans. $-\frac{1}{6}(2x+3)^{3/2}-\frac{3}{2}\sqrt{(2x+3)}$ (ii) $\int \frac{3x+2}{\sqrt{x+1}}dx$ Ans. $2(x+1)^{3/2}-2\sqrt{x+1}$

25. (i) $\int \frac{x+1}{\sqrt{x+3}}dx$ Ans. $\frac{3}{2}(x+3)^{3/2}-4\sqrt{(x+3)}$

(ii) $\int \frac{x^2 dx}{\sqrt{x+2}}$ Ans. $\frac{2}{5}(x+2)^{5/2}-\frac{8}{3}(x+2)^{3/2}+8(x+2)^{1/2}$

26. (i) $\int \frac{4x+3}{\sqrt{2x-1}}dx$ Ans. $\frac{2}{3}(2x-1)^{3/2}+5\sqrt{2x-1}$ (ii) $\int \frac{1-x}{\sqrt{1+x}}dx$ Ans. $4\sqrt{x-1}+\frac{2}{3}(x-1)^{3/2}$

(iii) $\int \frac{1-x}{\sqrt{1+x}}dx$ Ans. $4\sqrt{x-1}-\frac{2}{3}(1+x)^{3/2}$

27. $\int \frac{dx}{\sqrt{x}+x}$ Ans. $2\log(1+\sqrt{x})$

28. $\int \frac{dx}{\sqrt{x}+\sqrt{1+x}}dx$ Ans. $\frac{2}{3}\{(1+x)^{3/2}-x^{3/2}\}$

29. $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$ Ans. $\frac{2}{3}\{(x+1)^{3/2}+x^{3/2}\}$

30. $\int \frac{dx}{\sqrt{x+2}-\sqrt{x+3}}$ Ans. $\frac{-2}{3}\{(x+2)^{3/2}+(x+3)^{3/2}\}$

31. $\int \frac{dx}{\sqrt{x+1}+\sqrt{x+2}}$ Ans. $-\frac{2}{3}(x+1)^{3/2}-(x+2)^{3/2}$

3.5.3 STANDARD INTEGRALS

(A) $\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}(x>a)$

(B) $\dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a}\log\dfrac{a+x}{a-x}, (x < a)$

(c) $\dfrac{dx}{\sqrt{x^2 \pm a^2}} = \log\left(x + \sqrt{x^2 \pm a^2}\right)$

Example 152: $\dfrac{3dx}{x^2 - 1}$

$I = \dfrac{3dx}{x^2 - 1} = \dfrac{3}{(x+1)(x-1)} dx$

$= \dfrac{3}{2} \quad \dfrac{1}{x-1} - \dfrac{1}{x+1} \quad dx$

$= \dfrac{3}{2} \quad \dfrac{1}{x-1}\ dx - \dfrac{dx}{x+1}$

$= \dfrac{3}{2}\{\log(x-1) - \log(x+1)\}$

$= \dfrac{3}{2}\ \log\dfrac{x-1}{x+1}$

Example 153: $\dfrac{xdx}{x^4 - 1}$ Let $x^2 = u$, then $2xdx = du$

$\therefore \quad I = \dfrac{1}{2}\ \dfrac{du}{u^2 - 1} = \dfrac{1}{2}.\dfrac{1}{2}\log\dfrac{u-1}{u+1} = \dfrac{1}{4}\log\dfrac{x^2-1}{x^2+1}$

Example 154: $\dfrac{dx}{4 - x^2}$

$I = \dfrac{dx}{(2+x)(2-x)} = \dfrac{1}{4} \quad \dfrac{1}{2+x} + \dfrac{1}{2-x} \quad dx = \dfrac{1}{4}\ \dfrac{dx}{2+x} + \dfrac{1}{4}\ \dfrac{dx}{2-x}$

$= \dfrac{1}{4}\log(2+x) - \dfrac{1}{4}\log(2-x) = \dfrac{1}{4}\log\dfrac{2+x}{2-x}$

Alternative way. $\dfrac{dx}{2^2 - x^2} = \dfrac{1}{4}\log\dfrac{2+x}{2-x}$

Example 155: $\dfrac{x}{2x^4 - 3x^2 - 2}dx$ Let $x^2 = u$, $2xdx = du$

$I = \dfrac{1}{2}\ \dfrac{du}{2u^2 - 3u - 2} = \dfrac{1}{2}\ \dfrac{du}{(u-2)(2u+1)}$

$$= \frac{1}{2}.\frac{1}{5} \quad \frac{1}{u-2} - \frac{2}{2u+1} \; du = \frac{1}{10} \quad \frac{du}{u-2} - \frac{1}{10} \quad \frac{2du}{2u+1}$$

$$= \frac{1}{10}\log(u-2) - \frac{1}{5}\log(2u+1) = \frac{1}{10}\log(x^2-2) - \frac{1}{5}\log(2x^2+1) + c$$

Example 156: $\dfrac{x^3 dx}{\sqrt{x^8 \pm 1}}$

$$I = \frac{x^3 dx}{\sqrt{(x^4)^2 \pm 1}}$$ Let $x^4 = u$ or, $4x^3 dx = du$.

$$= \frac{1}{4} \quad \frac{du}{\sqrt{(u)^2 \pm 1}} = \frac{1}{4}\log\left(u + \sqrt{u^2 \pm 1}\right) \quad \text{(by C)}$$

$$= \frac{1}{4}\log\left(x^4 + \sqrt{x^8 \pm 1}\right).$$

SELF EXAMINATION QUESTIONS

Evaluate :

1. $\dfrac{x^2 dx}{x^6 - 1}$ Ans. $\dfrac{1}{6}\log\dfrac{x^3-1}{x^3+1}$

2. $\dfrac{1}{(\log x)^6 - 9}.\dfrac{dx}{x}$ Ans. $\dfrac{1}{6}\log\dfrac{\log x - 1}{\log x + 1}$

3. $\dfrac{dx}{2x^2 + 3x - 1}$ Ans. $\dfrac{1}{\sqrt{17}}\log\dfrac{4x+3-\sqrt{17}}{4x+3+\sqrt{17}}$

4. $\dfrac{x dx}{\sqrt{a^4 + x^4}}$ Ans. $\dfrac{1}{2}\log\left(x^2 + \sqrt{a^4 + x^4}\right)$

5. $\dfrac{dx}{\sqrt{16x^2 - 9}}$ Ans. $\dfrac{1}{4}\log\left(4x + \sqrt{16x^2 - 9}\right)$

6. (i) $\dfrac{dx}{x^2 + 2x - 1}$ Ans. $\dfrac{1}{2\sqrt{2}}\log\dfrac{x+1-\sqrt{2}}{x+1+\sqrt{2}}$

   (ii) $\dfrac{dx}{2x^2 - 4x - 7}$ Ans. $\dfrac{1}{6\sqrt{2}}\log \dfrac{\sqrt{2}(x-1)-3}{\sqrt{2}(x-1)+3}$

   (iii) $\dfrac{dx}{x^2 - 3x + 1}$ Ans. $\dfrac{1}{\sqrt{5}}\log\dfrac{2x-3-\sqrt{5}}{2x-3+\sqrt{5}}$

INTEGRATION BY PARTS

Integration of a Product:

Let u and v_1 be differential functions of x.

Then $\frac{d}{dx}(uv_1) = \frac{du}{dx}v_1 + u\frac{dv_1}{dx}$ (from diff. calculus)

Now integrating both sides w.r.t. x

We get $uv_1 = \quad \frac{du}{dx}v_1 \ dx + \quad u\frac{dv_1}{du} \ dx$

or, $\quad u\frac{dv_1}{dx} \ dx = uv_1 - \quad \frac{du}{dx}v_1 \ dx$ (transposing)

Taking $\frac{dv_1}{dx} = v$ then $v_1 = \ vdx$

The above result may be written as

$$(uv)dx = u \ vdx - \quad \frac{du}{dx} \ vdx \ dx$$

It states integral of product of two functions = 1st function (unchanged) × int. of 2nd – integral of (diff. 1st × int. of 2nd.).

Note : Care should be taken to choose properly the first function, i.e., the function not to be integrated.

Example 157: Evaluate $\quad xe^x dx$

$xe^x dx$ (here e^x is taken as second function)

$$= x \ e^x dx - \quad \frac{dx}{dx} \ e^x dx. \ dx = x.e^x - \ 1e^x dx = xe^x - e^x + c$$

Note : If e^x be taken as first function, integral becomes $\quad e^x xdx$

$$= e^x \ xdx - \ \frac{d}{dx}e^x. \ xdx \ dx = e^x.\frac{x^2}{2} - \ e^x.\frac{x^2}{2}dx$$

Now to find the value of $\quad e^x.\frac{x^2}{2}$ dx becomes complicated. So x is taken as first function for easy solution.

Example 158: $\int \log x \, dx = \quad \log x.1.dx$

$$= \log x \;\; dx - \quad \frac{d}{dx}\log x \;\; dx \;\; dx$$

$$= \log x.x - \quad \frac{1}{x}.x \;\; dx = x\log x - \quad dx = x\log x - x + c.$$

Example 159: Evaluate : $\int \log x^2 \, dx$

$$I = \;\; \log x^2.1.dx = \log x^2. \;\; 1.dx - \quad \frac{d}{dx}\log x^2. \;\; 1dx \;\; dx$$

$$= \log x^2.x - \quad \frac{2x}{x^2}.x \;\; dx = x\log x^2 - 2 \;\; dx = x\log x^2 - 2x + c$$

Example 160: Evaluate : $\int e^x (1 + x) \log (xe^x)dx$.

Let $xe^x = u$, $(e^x + xe^x)dx = du$ or, $e^x (1 + x) \, dx = du$.

Integral (i) = $\int \log u \, du = u \log u - u = xe^x \log(xe^x) - xe^x + c$

Example 161: Evaluate : $\int (x^2 - 2x + 5)e^{-x} \, dx$.

$I = \int x^2e^{-x}dx - 2 \int xe^{-x} \, dx + 5\int e^{-x} \, dx$

$$= x^2 \;\; e^{-x}dx - \quad \frac{d}{dx}x^2 \;\; e^{-x}dx \;\; dx - 2 \;\; xe^{-x}dx + 5e^{-x}(-1)$$

$$= x^2.e^{-x}(-1) - 2 \;\; xe^{x}.(-1)dx - 2 \;\; xe^{-x}dx - 5e^{-x}$$

$$= -x^2e^{-x} + 2 \;\; xe^{-x}dx - 2 \;\; xe^{-x}dx - 5e^{-x} = -x^2e^{-x} - 5e^{-x} + c$$

Standard Integrals :

1. $\sqrt{x^2 + a^2} \, dx = \dfrac{x\sqrt{x^2 + a^2}}{2} + \dfrac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right)$

2. $\sqrt{x^2 - a^2} \, dx = \dfrac{x\sqrt{(x^2 - a^2)}}{2} - \dfrac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right).$

Example 162: Find the value of $\sqrt{25x^2 + 16} \, dx$

$I = \sqrt{(5x)^2 + 4^2} \, dx.$ Let $5x = u$, $5dx = du$

$= \frac{1}{5}\sqrt{u^2+4^2}\,du = \frac{1}{5}\sqrt{u^2+4^2}\,du$

$= \frac{1}{5}\left[\frac{u\sqrt{u^2+4^2}}{2} + \frac{4^2}{2}\log\left(u+\sqrt{u^2+4^2}\right)\right]$ (by formula 1)

$= \frac{1}{5}\left[\frac{5x\sqrt{25x^2+16}}{2} + \frac{16}{2}\log\left(5x+\sqrt{25x^2+16}\right)\right]$

$= \frac{x\sqrt{25x^2+16}}{2} + \frac{8}{5}\log\left(5x+\sqrt{25x^2+16}\right) + c.$

SELF EXAMINATION QUESTIONS

Integrate :

1. x^2e^x. [Ans. $x^2e^x + 2e^x - 2xe^x$, c is added in every case]

2. xe^{4x} [Ans. $\frac{e^{4x}}{16}(4x-1)+c$]

3. xe^{ax}. [Ans. $\frac{e^{ax}}{a^2}(ax-1)+c$]

4. $x \log x$ [Ans. $\frac{x^2}{2}\log x - \frac{x^2}{4}+c$]

5. $x^3 e^x$. [Ans. $x^3 e^x - 3x^2e^x + 6xe^x - 6x^x$]

6. $(\log x)^2$. [Ans. $x (\log x)^2 - 2x \log x + 2x+c$]

7. $x (\log x)^2$. [Ans. $\frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4}+c$]

8. (i) $\frac{\log x}{x^2}$ Ans. $-\frac{\log x}{x} - \frac{1}{x}+c$

   (ii) $\frac{\log(1+x)}{(1+x)^2}dx$ Ans. $-\frac{\log x}{x} - \frac{1}{x}+c$

9. $(x + 1)^2 \log x$ Ans. $\frac{1}{3}\log x(x+1)^3 - \frac{1}{9}x^3 - \frac{x^2}{2} - x - \frac{1}{3}\log x + c$

10. $\frac{1}{2}\log(\log x)$. [Ans. $\log x \{\log(\log x)-1\}+c$]

11. $\log(x^2 + 2x + 1)$ [Ans. $x \log(x^2 + 2x + 1) - 2x + x\log(x + 1 + c)$]

12. $\log\left(x - \sqrt{x^2 - 1}\right)$. Ans. $x\log\left(x - \sqrt{x^2-1}\right) + \sqrt{x^2-1} + c$

13. $\log\left(x - \sqrt{x^2 + a^2}\right)$. Ans. $x\log\left(x - \sqrt{x^2+a^2}\right) - \sqrt{x^2+a^2} + c$

14. $e^x \; \frac{1}{x} - \frac{1}{x^2}$ Ans. $\frac{e^x}{x} + c$

15. $e^x \; \frac{1}{x+1} - \frac{1}{(x+1)^2}$ Ans. $\frac{e^x}{x+1} + c$

16. $\frac{1}{\log x} - \frac{1}{(\log x)^2}$ Ans. $\frac{x}{\log x} + c$

17. $\sqrt{x^2+9}$ Ans. $\frac{x\sqrt{x^2+9}}{2} + \frac{9}{2}\log\left(x + \sqrt{x^2+9}\right) + c$

18. $\sqrt{5 - 2x + x^2}$ Ans. $\frac{(x-1)\sqrt{5-2x+x^2}}{2} + 2\log\left\{(x-1) + \sqrt{5-2x+x^2}\right\} + c$

19. $\sqrt{x^2-4}$ Ans. $\frac{x}{2}\sqrt{x^2-4} - 2\log(x + \sqrt{x^2-4}) + c$

20. $\sqrt{4x^2 - 4x + 10}$. Ans. $\frac{2x-1}{4}\sqrt{4x^2-4x+10} + \frac{9}{4}\log\left\{2x - 1 + \sqrt{4x^2-4x+10}\right\} + c$

3.5.5 DEFINITE INTEGRALS

Definition:

Let a function f(x) has a fixed finite value in [a, b] for any fixed value of x in that interval i.e., for $a \le x \le$ and f(x) is continuous in [a, b], where a and b both are finite, (b>a).

Let the interval [a, b] be divided in equal parts having a length h. Now the points of division (on x axis) will be.

$x = a + h, a + 2h \ldots, a + (n-1)h, b - a = nh.$

Now $\lim_{h\to 0} h \;\; f(a+h) + f(a+2h) + \ldots\ldots + f(a+nh)]$

i.e., $\lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh)$, (if it exists) is called definite integral of the function f(x) between the limits a and b and is denoted symbolically by $\int_b^a f(x)dx$ i.e., $\int_b^a f(x)dx = \lim_{h \to 0} h \sum_{r=1}^{n} f(a+rh), b > a, b-a = nh.$

Note: (i) a is called as lower limit, while b is known as upper limit.

(ii) If a = 0, then $\int_a^b f(x)dx = \lim_{h \to 0} h \sum_{r=1}^{n} f\,(rh)$ here $nh = b$

(iii) If a = 0, b = 1, then $\int_0^1 f(x)dx = \lim_{h \to 1} h \sum_{r=1}^{n} f\,(rh)$ where $nh = 1$

(iv) if a > b, then $\int_a^b f(x)dx = -\int_a^b f(x)dx.$

(v) If a = b then $\int_a^b f(x)dx = 0$

Example 163: Evaluate the following definite integral from definition $\int_a^b 2dx.$

Here, f(x) = 2 (a constant) ∴ f(a + rh) = 2, nh = b – a.

Now from $\int_a^b f(x)dx = \lim_{h \to 0} h \sum_{r=1}^{n} f\,(a+rh)$ we find

$$\int_a^b 2dx = \lim_{h \to 0} h \sum_{r=1}^{n} 2 = \lim_{h \to 0} h.2n = \lim_{h \to 0} 2\,nh$$

$$= \lim_{h \to 0} 2(b-a), \text{ as } nh = b-a = 2(b-a)$$

Example 164: Evaluate from the first principle the value of $\int_a^b xdx.$

Here, f(x) = x ∴ f(a + rh) = a + rh, nh = b – a

Now from $\int_a^b f(x)dx = \lim_{h \to 0} h \sum_{r=1}^{n} f(a+rh),$ we get

$$\int_a^b xdx = \lim_{h \to 0} h \sum_{r=1}^{n} (a+rh) = \lim_{h \to 0} h\{na + h(1+2+3+\ldots+n)\}$$

$$= \lim_{h \to 0} h\left[na + h.\frac{n(n+1)}{2}\right] \quad ; \text{as } 1+2+\ldots+n = \frac{n(n+1)}{2}$$

$$= \lim_{h \to 0}\left[a(nh) + \frac{(nh)(nh+h)}{2}\right]$$

$$= \lim_{h \to 0} h \quad a(b-a) + \frac{(b-a)(b-a+h)}{2}$$

$$= a(b-a) + \frac{(b-a)^2}{2} = (b-a) \quad a + \frac{b-a}{2}$$

$$= (b-a) + \frac{(b+a)}{2} = \frac{1}{2}\left(b^2 - a^2\right)$$

Example 165: By the method of summation, find the value of $\int_0^1 (3x+5)dx$

Here, f(x) = (3x+5); a = 0, b = 1;

f(a + rh) = 3 (a + rh) + 5 = 5 +3rh, nh = b – a = 1 – 0 = 1

$$\therefore \int_0^1 (3x+5)dx = \lim_{h \to 0} h \sum_{r=1}^{n} f(a+rh) = \lim_{h \to 0} h \sum_{r=1}^{n} (5+3rh)$$

$$= \lim_{h \to 0} h\{5n + 3h(1+2+3+\ldots+n)\} = \lim_{h \to 0} h \quad 5n + 3h.\frac{n(n+1)}{2}$$

$$= \lim_{h \to 0} \quad 5.nh \frac{3}{2}(nh)(nh+h) \quad = \lim_{h \to 0} \quad 5.1 + \frac{3}{2}.1.(1+h)$$

$$= 5.1 + \frac{3}{2}.1.1 = 5 + \frac{3}{2} = \frac{10+3}{2} = \frac{13}{2}.$$

SELF EXAMINATION QUESTIONS

1. $\int_a^b dx$ [Ans. b – a]
2. $\int_a^b 10dx.$ [Ans. 10(b–a)]
3. $\int_a^b x^3dx$ [Ans. $\frac{1}{4}(b^4 - a^4)$]
4. $\int_0^1 dx$ [Ans. 1]
5. $\int_0^1 xdx$ [Ans. $\frac{1}{2}$]
6. $\int_0^1 x^2dx$ [Ans. $\frac{1}{3}$
7. $\int_1^2 5x^2dx$ [Ans. $\frac{35}{3}$]
8. $\int_0^1 (2x+5)dx$ (Ans. 6]

Definite Integral

In the previous part $\int_a^b f(x)\,dx$ has been defined as a limit of a sum. There is an important theorem in Integral Calculus known as *Fundamental theorem of Integral Calculus* which states :

If there exists a function ϕ (x) such that $\frac{d}{dx}\phi(x) = f(x)$ for every x in $a \le x \le b$ and if $\int_a^b f(x)dx$ exists, then $\int_a^b f(x)dx = \phi(b) - \phi(a)$.

$\int_a^b f(x)dx$, is read as 'Integral from a to b of f(x) dx' where a is lower limit and b is the upper limit.

Symbol : $\phi(b) - \phi(a)$ is written as $\left[\phi(x)\right]_a^b$, which is read as ϕ (x) from a to b.

$$\therefore \quad \int_a^b f(x)\,dx = \left[\phi(x)\right]_a^b = \phi(b) - \phi(a).$$

Reason for the name 'definite integral'.

Let us take f(x) dx = ϕ (x) + c instead of $\int$ f(x) dx = ϕ (x).

$$\text{Now} \int_a^b f(x)\,dx = \left[\phi(x) + c\right]_a^b = \left[\phi(b) + c\right] - \left[\phi(a) + c\right] = \phi(b) - \phi(a).$$

Here the arbitrary constant 'c' is absent and $\int_a^b f(x)dx = \phi(b) - \phi(a)$ and hence it is known as definite integral or in other words definite integral is unique.

Rule to Evaluate : $\int_a^b f(x)dx$. (i) Find the value of f(x) dx, leaving the arbitary constant.

(ii) In the value obtained, put x = b (upper limit) and x = a (lower limit).

(iii) Deduct the second value from the first value (after putting the values of x).

(iv) The result thus obtained will be the required value of the definite integral.

A Few Results :

1. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, k is constant

Example 166: $\int_0^1 2x\,dx = 2\int_0^1 x\,dx.$

2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Example 167: $\int_0^1 (x^2 \pm 2x)\,dx = \int_0^1 x^2 dx \pm 2\int_0^1 x\,dx.$

3. $\int_a^b f(x) = -\int_b^a f(x)\,dx$

i.e., interchange of limits indicate the change of sign.

Example 168: $\int_1^2 (x+1)\,dx = -\int_2^1 (x+1)\,dx.$

4. $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$ where $a < c < b$.

Example 169: $\int_0^2 x\,dx = \int_0^1 x\,dx + \int_1^2 x\,dx,$ as $0 < 1 < 2$.

SOLVED EXAMPLES

Example 170: (i) $\int_1^2 x\,dx$ (ii) $\int_0^1 x^4\,dx$ (iii) $\int_4^9 \sqrt{x}\,dx$ (iv) $\int_3^5 \frac{dx}{x}$ (v) $\int_1^2 e^{2x}dx.$

(i) $\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = \frac{4}{2} - \frac{1}{2} = 2 - \frac{1}{2} = \frac{3}{2}.$

(ii) $\int_0^1 x^4\,dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5} - \frac{0}{5} = \frac{1}{5} - 0 = \frac{1}{5}.$

(iii) $\int_4^9 \sqrt{x}\,dx = \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_4^9 = \frac{2}{3}\left[x^{3/2}\right]_4^9 = \frac{2}{3}.9^{3/2} - \frac{2}{3}4^{3/2}$

$= \frac{2}{3}\left(3^{2.3/2} - 2^{2.3/2}\right) = \frac{2}{3}\left(3^3 - 2^3\right) = \frac{2}{3}(27-8) = \frac{2.19}{3} = \frac{38}{3}.$

(iv) $\int_3^5 \frac{dx}{x} = \left[\log x\right]_3^5 = \log 5 - \log 3 = \log\frac{5}{3}.$

(v) $\int_1^2 e^{2x}dx = \left[\frac{e^{2x}}{2}\right]_1^2 = \frac{1}{2}\left(e^{2.2} - e^{2.1}\right) = \frac{1}{2}\left(e^4 - e^2\right).$

Example 171: Evaluate : $\int_1^2 \frac{x^2+2x+5}{x} dx.$

$$I = \int_1^2 \left(x+2+\frac{5}{x}\right) dx = \left[\frac{x^2}{2}+2x+5\log x\right]_1^2$$

$$= \left(\frac{2^2}{2}+2.2+5\log 2\right) - \left(\frac{1}{2}+2+5\log 1\right)$$

$$= (6+5\log 2) - \frac{5}{2} \quad (\text{as } \log 1 = 0) = \frac{7}{2}+5\log 2.$$

Example 172: Evaluate : $\int_0^1 \frac{1-x}{1+x} dx.$

$$\int \frac{1-x}{1+x} dx = \int \left(\frac{2}{1+x}-1\right) dx = 2\int \frac{dx}{1+x} - \int dx = 2\log(1+x) - x$$

$$\therefore I = \left[2\log(1+x) - x\right]_0^1 = (2\log 2 - 1) - 2\log 1 = 2\log 2 - 1$$

Example 173: Evaluate : $\int_1^2 \left(1+\frac{2}{\sqrt{x}}+3x\right) dx$

$$\int \left(1+\frac{2}{\sqrt{x}}+3x\right) dx = \int dx + 2\int x^{-\frac{1}{2}} dx + 3\int x dx$$

$$= x + 2\frac{x^{1/2}}{1/2} + \frac{3x^2}{2} = x + 4\sqrt{x} + \frac{3x^2}{2}$$

$$\therefore \quad I = \left[x+4\sqrt{x}+\frac{3x^2}{2}\right]_1^2 = \left(2+4\sqrt{2}+\frac{3}{2}.4\right) - \left(1+4+\frac{3}{2}\right) = \frac{3+8\sqrt{2}}{2}.$$

Example 174: Evaluate : $\int_0^1 x^2 e^x dx$

$$\int x^2e^x dx = x^2e^x - \int 2xe^x dx = x^2e^x - 2(xe^x - e^x)$$

$= e^x (x^2 - 2x + 2)$, (integrating by parts)

$$I = \left[e^x(x^2-2x+2)\right]_0^1 = e(1-2.1+2) - e^0(0-0+2) = e-2.$$

SELF EXAMINATION QUESTIONS

Evaluate :

1. (i) $\left[\frac{x^2}{2}\right]_0^1$ (ii) $\left[x^2+4x\right]_1^2$ (iii) $\left[\frac{x^2}{3}+2x\right]_0^1$ [Ans. (i) $\frac{1}{2}$, (ii) 7, (iii) $\frac{7}{3}$]

2. (i) $\int_0^1 dx$ (ii) $\int_2^3 x dx$ (iii) $\int_0^1 2x^4\, dx$ (iv) $\int_0^4 x^2\, dx$ (v) $\int_1^6 \frac{x}{6}\, dx$.

[Ans. (i) 1, (ii) $\frac{5}{2}$, (iii) $\frac{2}{5}$, (iv) $\frac{64}{3}$, (v) $\frac{35}{12}$]

3. (i) $\int_0^1 (2x+3)\, dx$ (ii) $\int_{-1}^1 (x+1)^2\, dx$ (iii) $\int_{-2}^2 (x+2)^3\, dx$

(iv) $\int_0^2 (x^2-x+1)dx$. [Ans. (i) 4, (ii) $\frac{8}{3}$, (iii) 64, (iv) $\frac{8}{3}$,]

4. (i) $\int_6^7 \frac{dx}{x-4}$ (ii) $\int_2^8 \frac{dx}{2x+3}$ (iii) $\int_1^2 \frac{dx}{ax+b}$ (iv) $\int_0^1 \frac{1}{(2x+1)^2}dx$.

[Ans. (i) $\log\frac{3}{2}$, (ii) $\frac{1}{2}\log\frac{19}{7}$ (iii) $\frac{1}{a}\ \log(2a+b)-\log(a+b)$ (iv) $\frac{1}{3}$]

5. (i) $\int_0^9 \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right) dx$ (ii) $\int_0^4 \frac{5x^2-3x+7}{\sqrt{x}}\, dx$. [Ans. (i) 12, (ii) 76]

(iii) $\int_1^2 \left(x\sqrt{x}+\frac{2}{x}-\frac{5}{(x^{3/2})}\right)dx$ [Ans. $\frac{1}{5}\left(58-42\sqrt{2}\right)$]

6. (i) $\int_0^2 e^x dx$ (ii) $\int_a^b e^{mx} dx$ (iii) $\int_0^2 e^{-x}\, dx$. [Ans. (i) e^2-1, (ii) $\frac{1}{m}\left(e^{mb}-e^{ma}\right)$ (iii) $1-e^{-2}$]

7. (i) $\int_1^2 \log x\, dx$ (ii) $\int_1^2 x\log x\, dx$. (iii) $\int_0^1 x\log(1+2x)dx$.

[Ans. (i) 2 log 2 – 1, (ii) $2\log 2-\frac{3}{4}$ (iii) $\frac{3}{8}\log 3$]

8. (i) $\int_1^e x\log x\, dx$ (ii) $\int_0^1 x\log(x+2)dx$. [Ans. (i) $\frac{e^2}{4}+\frac{1}{4}$, (ii) $2\log 2-\frac{3}{2}\log 3+\frac{3}{4}$]

9. (i) $\int_1^2 xe^x\, dx$ (ii) $\int_0^1 xe^x\, dx$

(iii) $\int_0^1 x^2e^x\, dx$ (iv) $\int_1^2 x^2e^x\, dx$. (v) $\int_0^1 x^2e^{3x}\, dx$.

[Ans. (i) e^2, (ii) 1, (iii) e – 2, (iv) $2e^2-e$ (v) $\frac{1}{27}\left(5e^3-2\right)$]

3.5.6 METHOD OF SUBSTITUTION

Rule of evaluate $\int_a^b f(x)\,dx$ by the substitution x = f (u) :

1. In the integral put x = ϕ (u) and dx = ϕ' (u) and dx = ϕ' (u) du.

2. From the relation x = f (u),

For x = a, find the corresponding value of u say α.

For x = b, find the corresponding value of u, say β.

3. Evaluate the new integrand with the new limits the value thus obtained will be the required value of the original integrand.

Note : In a definite integral substitution is reflected in three places :

(i) in the integrand, (ii) in the differential, and (iii) in the limits.

This idea will be clear from the following examples.

Example 175: Evaluate : $\int_0^1 \frac{xdx}{\sqrt{1+x^2}}$

Let $1 + x^2 = u^2$ when x = 1, $u^2 = 1 + 1 = 2$ or, $u = \sqrt{2}$

or, 2xdx = 2udu when x = 0, $u^2 = 1 + 0 = 1$ or, u = 1

or, xdx = udu

$$\therefore\ I = \int_1^{\sqrt{2}} \frac{udu}{\sqrt{u^2}} = \int_1^{\sqrt{2}} \frac{udu}{u} = \int_1^{\sqrt{2}} du = [u]_1^{\sqrt{2}} = \sqrt{2} - 1$$

Example 176: $\int_0^1 \frac{x^7dx}{1+x^8}$. Let $1 + x^8 = u$, $8x^7\,dx = du$

When x = 1, u = 1+1 = 2 ; x = 0, u = 1 + 0 = 1

$$\therefore\ I = \frac{1}{8}\int_1^2 \frac{du}{u} = \frac{1}{8}[\log u]_1^2 = \frac{1}{8}(\log 2 - \log 1) = \frac{1}{8}\log 2.$$

Example 177: $\int_a^b \log x \frac{dx}{x}$. Let log x = u, $\frac{dx}{x} = du$, for x = b, u = log b

x = a, u = log a

$$\therefore\ I = \int_{\log a}^{\log b} udu = \left[\frac{u^2}{2}\right]_{\log a}^{\log b} = \frac{1}{2}\left[(\log b)^2 - (\log a)^2\right]$$

$$= \frac{1}{2}(\log b + \log a)(\log b - \log a) = \frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right).$$

Example 178: $\int_1^2 \frac{x^2-1}{x^2}\, e^{x+1/x}\, dx.$

Let $x+\frac{1}{x}=u$ or, $\left(1-\frac{1}{x^2}\right) dx = du$ or, $\frac{x^2-1}{x^2}\, dx = du$

For $x = 2,\ u = 2+\frac{1}{2}=\frac{5}{2}$; $x = 1,\ u = 1 + 1 = 2$

$\therefore I = \int_2^{5/2} e^u du = \left[e^u\right]_2^{5/2} = e^{5/2} - e^2.$

Example 179: Evaluate : $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}}$

$$\frac{dx}{\sqrt{x+1}+\sqrt{x}} = \frac{dx(\sqrt{x+1}-\sqrt{x})}{(\sqrt{x+1}+\sqrt{x})} = (\sqrt{x+1}-\sqrt{x})dx$$

$$=\frac{2}{3}(x+1)^{3/2}-\frac{2}{3}x^{3/2}$$

$$\therefore I = \frac{2}{3}\left[(x+1)^{3/2}-x^{3/2}\right]_0^1 = \frac{2}{3}\left[\left(2^{2/3}-1\right)-1\right] = \frac{2}{3}(2^{3/2}-2) = \frac{4}{3}\left(\sqrt{2}-1\right).$$

SELF EXAMINATION QUESTIONS

Find the value of :

1. $\int_0^2 \frac{x^2\, dx}{\sqrt{1+x^3}}.$ [Ans. $\frac{4}{3}$]

2. $\int_0^2 x\sqrt{4-x^2}\, dx.$ [Ans. $\frac{8}{3}$]

3. $\int_0^2 \frac{6x+5}{3x^2+5x+1} dx.$ [Ans. log 23]

4. $\int_0^3 \sqrt{x+1}\, dx.$ [Ans. $\frac{14}{3}$]

5. $\int_0^{1/3} \sqrt{1-3x}\, dx.$ [Ans. $\frac{2}{9}$]

6. $\int_0^1 x^3\sqrt{1+3x^4}\, dx.$ [Ans. $\frac{7}{18}$]

7. $\int_1^2 x^2\sqrt{x^3+8}\, dx.$ [Ans. $\frac{74}{9}$]

8. (i) $\int_1^{e^2} \frac{dx}{x\,(1+\log x)}.$ [Ans. log 3] (ii) $\int_e^2 \frac{dx}{x\log x}.$ [Ans. log (log 2)]

9. $\int_1^{e^2} \frac{dx}{x\,(1+\log x)^2}.$ [Ans. $\frac{2}{3}$]

10. (i) $\int_0^1 \frac{dx}{\sqrt{x}+x}$. [Ans. 2 log 2] (ii) $\int_0^a \frac{xdx}{\sqrt{a^2-x^2}}$. [Ans. a]

Show that :

11. $\int_0^{\log 2} \frac{e^x\,dx}{e^x+1} = \log\frac{3}{2}$.

12. $\int_0^2 x(x-1)(x-2)dx = 0$.

13. $\int_0^{1/2} \frac{dx}{\sqrt{3-2x}} = \sqrt{3}-\sqrt{2}$.

14. (i) $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}} = \frac{4}{3}\sqrt{2}$.

(ii) $\int_0^3 \frac{x\,dx}{\sqrt{x+1}+\sqrt{5x+1}} = \frac{14}{15}$.

15. $\int_1^e \frac{dx}{x(1+\log x)^2} = \frac{1}{2}$.

16. $\int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx = e - \frac{2}{\log 2}$.

Evaluate :

17. $\int_0^1 \frac{xe^x\,dx}{(x+1)^2}$. [Ans. $\frac{e}{2}-1$]

18. $\int_2^3 \frac{x^5\,dx}{x^4-1}$. [Ans. $\frac{5}{2}+\frac{1}{4}\log\frac{4}{3}$]

[Hints : $x^2 = u, \frac{1}{2}\int \frac{u^2du}{u^2-1} = \frac{1}{2}\int\left(1+\frac{1}{u^2-1}\right)du = \frac{1}{2}u+\frac{1}{4}\log\frac{u-1}{u+1}$ & etc.]

Summation of a Series by Definite Integral :

From the definition of definite integral, we know

$$\int_a^b f(x)dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh), \text{ where } nh = b-a.$$

For a = 0 and b = 1, we find nh = 1 – 0 = 1 or $h = \frac{1}{n}$

If now $h \to 0+$, then $n \to \infty$.

$$\therefore \int_0^1 f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right).$$

Example 180: Evaluate : $\lim_{n\to\infty}\frac{1+2^{10}+3^{10}+...+n^{10}}{n^{10}}$

Given expression $= \lim_{n\to\infty}\frac{1}{n}\left[\frac{1^{10}+2^{10}+3^{10}+...+n^{10}}{n^{10}}\right]$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^{10}+\left(\frac{2}{n}\right)^{10}+\left(\frac{3}{n}\right)^{10}+...+\left(\frac{n}{n}\right)^{10}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{10} = \int_0^1 x^{10}\,dx = \left[\frac{x^{11}}{11}\right]_0^1 = \frac{1}{11}-0=\frac{1}{11}.$$

Example 181: Evaluate : $= \lim_{n\to\infty}\left[\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+\frac{3^2}{3^3+n^3}+...+\frac{n^2}{n^3+n^3}\right]$

Given expression $= \lim_{n\to\infty}\frac{1}{n}\left[\frac{1^2.n}{1^3+n^3}+\frac{2^2.n}{2^3+n^3}+...+\frac{n^2n}{n^3+n^3}\right]$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(\frac{1}{n}\right)^2}{1+\left(\frac{1}{n}\right)^3}+\frac{\left(\frac{2}{n}\right)^2}{1+\left(\frac{2}{n}\right)^3}+...+\frac{\left(\frac{n}{n}\right)^2}{1+\left(\frac{n}{n}\right)^3}\right]$$

[dividing each term of numertator and denominator by n^3]

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3} = \int_0^1\frac{x^2}{1+x^3}\,dx.$$ [Put $1 + x^3 = u$. $3x^2\,dx = du$ and etc.]

$$= \left[\frac{1}{3}\log\left(1+x^3\right)\right]_0^1 = \frac{1}{3}(\log 2-\log 1)=\frac{1}{3}\log 2.$$

Example 182: Evaluate : $\underset{h\to\infty}{Lt}\left[\frac{1}{1+2n}+\frac{1}{2+2n}+....+\frac{1}{n+2n}\right]$

$$\text{Expression} = \underset{h\to\infty}{Lt}\left[\frac{1/n}{1/n+2n/n}+\frac{1/n}{2/n+2n/n}+....+\frac{1/n}{n/n+2n/n}\right]$$

$$= \underset{h\to\infty}{Lt}\frac{1}{n}\left[\frac{1}{1/n+2}+\frac{1}{2/n+2}+...+\frac{1}{n/n+2}\right] = \underset{h\to\infty}{Lt}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\frac{r}{n}+2} = \int_0^1 \frac{dx}{x+2}$$

$$= \left[\log(x-2)\right]_0^1 = \log_e 3 - \log_e 2 = \log_e \frac{3}{2}.$$

SELF EXAMINATION QUESTIONS

Evaluate :

1. $\lim\limits_{n\to\infty}\frac{1^2+2^2+3^2+...+n^2}{n^3}$ [Ans. $\frac{1}{3}$]

2. $\lim\limits_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}...+\frac{1}{n+n}\right]$ [Ans. log 2]

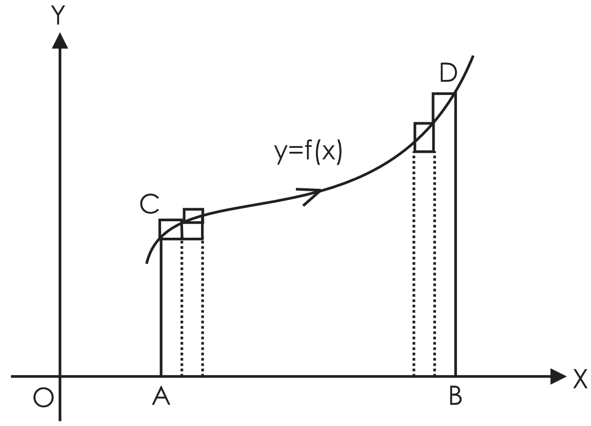
3. $\lim\limits_{n\to\infty}\left[\frac{1}{n+m}+\frac{1}{n+2m}+...+\frac{1}{n+nm}\right]$ [Ans. $\frac{1}{m}\log(1+m)$]

4. $\lim\limits_{n\to\infty}\frac{1^m+2^m+3^m+...+n^m}{n^{m+1}}$. [Ans. $\frac{1}{m+1}$]

5. $\lim\limits_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+....+\frac{1}{8n}\right]$ [Ans. $\frac{3}{8}$].

3.5.7 GEOMETRICAL INTERPRETATION OF A DEFINITE INTEGRAL

$\int_a^b f(x)\,dx.$ Let f(x) be a function continuous in [a, b], where a and b are fixed finite numbers, (b > a). Let us assume for the present f(x) is positive for $a \le x \le b$. As x increases from a to b, values of f(x) also increases.



In the figure the curve CD represents the function f(x), OA = a, OB = b, Ac = f(a) and BD = f(b).

Let S represent the area bounded by the curve y = f(x), the x-axis and the ordinates corresponding to x = a and x = b.

Divide [a, b], i.e., part AB into n finite intervals each of length h so that nh = b – a or a + nh = b.

Let S_1 = sum of rectangles standing on AB and whose upper sides lie every where below the curve y = f(x). and S_2 = sum of rectangles, whose upper sides lie above the curve y = f (x).

Now $S_1 = hf(a) + hf(a + h) + \ldots + hf(a + \overline{nh})$

$$= h\sum_{r=1}^{n} f(a+rh) + hf(a) - hf(a+nh)$$

$$= h\sum_{1}^{n} f(a+rh) + hf(a) - hf(b), \ldots (1), \quad \text{as } a + nh = b$$

and $\quad S_2 = hf(a+h) + h(a+2h) + \ldots + hf(a+nh)$

$$= h\sum_{r=1}^{n} f(a+rh)$$

From the figure, it is now clear that $S_1 < S < S_2$. $\quad\ldots(2)$

From nh = b – a, we get $h = \frac{b-a}{n}$, so as $n \to \infty$, $h \to 0$

Since f(a) and f(b) are finite numbers,

So hf (a) $\to$ 0, and hf (b) $\to$ 0 as h $\to$ 0.

From Eq. (1) we get

$$\lim_{h\to 0} S_1 = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh) + \lim_{h\to 0} hf(a) - \lim_{h\to 0} hf(b)$$

$$= \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh) + 0 - 0 = \int_a^b f(x)dx$$

Similarly, From Eq. (2),

$$\lim_{h\to 0} S_2 = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh) = \int_a^b f(x)dx. \text{ when } h \to 0,$$

$S_1 \to \int_a^b f(x)dx$ and $S_2 \to \int_a^b f(x)dx$. But $S_1 < S < S_2$:

$\therefore \quad S = \int_a^b f(x)dx.$

So the definite integral $\int_a^b f(x)dx$ geometrically represents the area enclosed by the curve y = f(x), the x-axis and the ordinates the x = a and x = b.

Observation :

1. If the values of f(x) decrease gradually corresponding to the increasing values of x, then also it may be shown similarly that S = $\int_a^b f(x)dx$.

2. If f(x) be continuous and positive in [a, b] and f(x) is increasing in [a, c], and f(x) is decreasing in [c, b], where a < c < b, then

$$\int_a^c f(x)\,dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

Steps to set up a proper definite integral corresponding to a disired area :

1. Make a sketch of the graph of the given function.

2. Shade the region whose area is to be calculate.

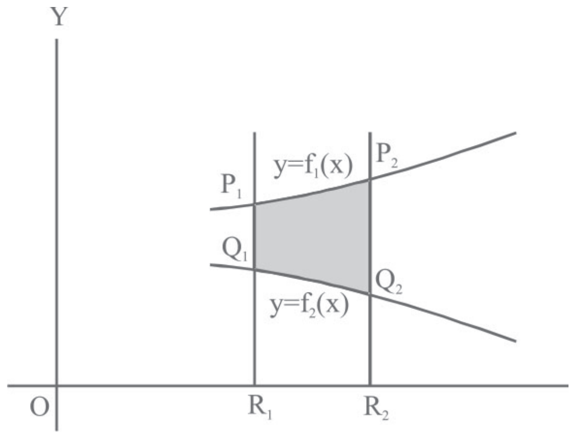
3. In choosing the limits of integration, the smaller value of x at ordinate is drawn will be taken as lower limit and the greater as upper limit (i.e., we are to move from left to right on x-axis) and then to evaluate the definite integral.

4. Only the numerical value (and not the algebraic value) of the area will be considered, i.e., we will discard the –ve sign, if some area comes out to be –ve (after calculation).

5. If the curve is symmetrical, then we will find, area of one symmetrical portion and then multiply it by n, if there are n symmetrical portions.

Area between two given curves and two given ordinates :

Let the area be bounded by the given curves y = f_1 (x) and y = f_2(x) and also by two given or dinates x = a and x = b, and is indicated by $p_1q_1q_2p_2p_1$ (refer the figure). Here OR_1 = a and OR_2 = b.



Now area $P_1 Q_1 Q_2 P_2 P_1$ = area $P_1 R_1 R_2 P_2$ – area $Q_1 R_1 R_2 Q_2$

$$= \int_a^b f_1(x)dx - \int_a^b f_2(x)\,dx$$

$$= \int_a^b \{f_1(x) - f_2(x)\}dx$$

$$= \int_a^b (y_1 - y_2)dx,$$

where y_1 and y_2 are the ordinates of the two curves $P_1 P_2$ and $Q_1 Q_2$ corresponding to the same abscissa x.

Some Well-known Curves :

It is expected that students are already acquinted with the following well-known curves :

1. Straight line : $ax + by + c = 0$

2. Circle : $x^2 + y^2 = a^2$

3. Parabola : $y^2 = 4ax$

4. Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$

5. Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

Section - B
STATISTICS

Study Note - 4

STATISTICAL REPRESENTATION OF DATA

This Study Note includes

4.1 DIAGRAMATIC REPRESENTATION OF DATA

4.1.1. Data :

A statistician begins the work with the collection of data i.e. numerical facts. The data so collected are called *raw materials* (or *raw data*). It is from these raw materials, a statistician analysis after proper classification and tabulation, for the final decision or conclusion. Therefore it is undoubtedly important that the raw data collected should be clear, accurate and reliable.

Before the collection of data, every enquiry must have a definite object and certain scope, that is to say, what information will be collected for whom it will be collected, how often or at what periodically it will be collected and so on. If the object and the scope of enqiry are not clearly determined before hand, difficulties may arise at the time of collection which will be simply a wastage of time and money.

4.1.2. Statistical Units :

The unit of measurement applied to the data in any particular problem is the statistical unit.

Physical units of the measurement like quintal, kilogramme, metre, hour and year, etc. do not need any explanation or definition. But in some cases statistician has to give some proper definition regarding the unit. For *examples,* the wholesale price of commodity. Now what does the form 'wholesale price' signify ? Does it stand for the price at which the producer sells the goods concrened to the stockist, or the price at which the stockist sells to a wholesaler ? Is it the price at which the market opened at the day of enquiry ? Many such problems may arise as stated. It is thus essential that a statistician should define the units of data before he starts the work of collection.

4.1.3. Types of Methods of Collection of Data :

Statistical data are usually of two types :

(i) Primary, (ii) Secondary

Data which are collected for the first time, for a specific purpose are known as *primary data,* while those used in an investigation, which have been originally collected by some one else, are known as *secondary data.*

For *example,* data relating to national income collected by government are primary data, but the same data will be secondary while those will be used by a different concern.

Let us take another example, known to everyone. In our country after every ten years counting of population is done, which is commonly known as Census. For this data are collected by the Government of India. The data collected are known as primary data. Now in the data, except population information about age of persons, education, income etc. are available. Now a separate department of the government or any other private concern use these related data for any purpose, then the data will be known as secondary data to them.

Data are primary to the collector, but secondary to the user.

Example.

For primary data :

(i) Reserve Bank of India Bulletin (monthly)

(ii) Jute Bulletin (monthly), (published by Govt. of India).

(iii) Indian Textile Bulletin (monthly).

(vi) Statement of Railway Board (yearly), (published by Ministry of Railway, Govt. of India).

For secondary data :

(i) Statistical Abstract of the Indian Union

(ii) Monthly Abstract of Statictics.

(iii) Monthly Statistical Digest.

(iv) International Labour Bulletin (monthly).

4.1.3.1. Primary Method :

The following methods are common in use :

(i) **Direct Personal Observation :** Under this method, the investigator collects the data personally. He has to go to the spot for conducting enquiry has to meet the persons concerned. It is essential that the investigator should be polite, tactful and have a sense of observation.

This method is applicable when the field of enquiry is small and there is an intention of greater accuracy. This method however, gives satisfactory result provided the investigator is fully dependable.

(ii) **Indirect Oral Investigation :** In this method data are collected through indirect sources. Persons having some knowledge regarding the enquiry are cross-examined and the desired information is collected. Evidence of one person should bot be relied, but a number of views should be taken to find out real position. This method is usually adopted by enquiry committees or commissions appointed by governments or semi-government or private institutions.

(iii) **Schedules and Questionnaires :** A list of questions regarding the enquiry is prepared and printed. Data are collected in any of the following ways :

(a) *By sending the questionnaire to the persons concerned with a request to answer the questions and return the questionnaire.*

(b) *By sending the questionnaire through enumerators for helping the informants.*

(iv) **Local Reports :** This method does not imply a formal collection of data. Only local agents or correspondents are requested to supply the estimate required. This method gives only approximate results, of course at a low cost.

4.1.3.2. Secondary Method :

The main sources from which secondary data are collected are given below–

(i) Official publications by the Central and State Government, District Boards,

(ii) Reports of Committees, Commissions.

(iii) Publications by Research Institutions, Universities,

(iv) Economic and Commercial Journals.

(v) Publications of Trade Associations, Chambers of Commerce, etc.

(vi) Market reports, individual research works of Statisticians.

Secondary data are also available from unpublished records of government offices, chambers of commerce, labour bureaus, etc.

4.1.3.2.1. Editing and Scrutiny :

Secondary data should be used only after careful enquiry and with due criticism. It is advisable not to take them at their face value. Scrutiny is essential because the data might be inaccurate, unsuitable and inadequate. According to Bowley, "It is never safe to take published statistics at their face value without knowing their meanings and limitations ...."

4.1.3.2.2. Universe or Population :

Statistics is taken in relation to a large data. Single and unconnected data is not statistics. In the field of any statistical enquiry there may be persons, items or any other similar units. The aggregate of all such similar units under consideration is called Universe or Population.

That is, for collecting the data regarding height, weight or age of the male candidates who appeared in the last H.S. Examination, the aggregate of such candidates is universe. Universe may be aggregate of items or any other similar things other than persons. The books in your college library or produced goods in a factory may be taken as Universe.

Population may be finite or infinite according to finite or infinite number of members. In the field of enquiry if the number of units is finite, then Population or Universe is finite. For Example, first class cricket or football players in India is finite. But the temperature in any day at Calcutta is infinite, although temperature lies between two finite limits. Within these two finite limits it takes up an infinity of values.

4.1.3.2.3. Sample :

If a part is selected out of the Universe then the selected part (or portion) is sample. It means sample is a part of the Universe.

So, suppose the screws or bulbs produced in a factory are to be tested. The aggregate of all such items is universe but it is not possible to test every item. So in such case, a part of the whole i.e., universe is taken and then tested. Now this part is known as sample.

Note. While collecting primary data (discussed before) it should be decided at first whether the purpose will be solved if collection is made from universe or sample.

4.1.4. Clasification and Tabulation

4.1.4.1. Classification :

It is the process of arranging data into different classes or group according to resemblance and similarities. An ideal classification should be unambiguous, stable and flexible.

Type of Classification :

There are two types of classification depending upon the nature of data.

(i) Classification according to attribute – if the data is of a descriptive nature having several qualifications i.e. males, female, illiterate, etc.

(ii) Classification according to class-interval if the data are expressed in numerical quantities i.e... ages of person vary and so do their heights and weights.

Classification according to Attributes :

(i) Simple classification is that when one attribute is present i.e. classification of persons according to sex –males or female.

(ii) Manifold classification is that when more than one attributes are present simultaneously two attributes –deafness and sex. A person may be either deaf or not deaf, further a person may be a male or a female. The data, thus are to be divided into four classes :-

(a) males who are deaf,

(b) males who are not deaf,

(c) females who are deaf,

(d) females who are not deaf.

The study can be further continued, if we find another attribute, say religion.

Classification according to Class-intervals :

The type arises when direct measurements of data is possible. Data relating to height, weight, production etc. comes under this category. For instance persons having weight, say 100-110 Lbs, can form one group, 110-120 lbs. another group and so on. In this way data are divided into different classes ; each of which is known as class interval. Number of items which fall in any class-interval is known as class frequency. In the class-intervals mentioned above, the first figures in each of them are the lower limits, while the second figure are the upper limits. The difference between the limits of a class interval is known as magnitude of the class interval. If for each class intervals the frequencies given are aggregates of the preceding frequencies, they are known as cumulative frequencies. The frequencies may be cumulated either from top or from below.

DISCRETE AND CONTINUOUS SERIES :

Statistical series may be either discrete or continuous. A discrete series is formed from items which are exactly measurable, Every unit of data is separate, complete and not capable of divisions. For instance, the number of students obtaining marks exactly 10, 14, 18, 29, can easily be counted. But phenomenon like height or weight cannot be measured exactly or with absolute accuracy. So the number of students (or individuals) having height exactly 5’ 2” cannot be counted. Exact height may be either 5’2” by a hundredth part of an inch. In such cases, we are to count the number of students whose heights lie between 5’ 0” to 5’ 2”. Such series are known as ‘continuous’ series.

Example 1:

Discrete Series		Continuous Series	
Marks	**No. of Students**	**Height (inch)**	**No. of students**
10	12	58 – 60	6
14	16	60 – 62	10
18	15	62 – 64	13
20	7	64 – 66	11

4.1.4.2. TABULATION :

Tabulation is a systematic and scientific presentation of data in a suitable form for analysis and interpretation.

After the data have been collected, they are tabulated i.e. put in a tabular form of columns and rows. The function of tabulation is to arrange the classified data in an orderly manner suitable for analysis and interpretation. Tabulation is the last stage in collection and compilation of data, and is a kind of stepping-stone to the analysis and interpretation.

A table broadly consists of five parts –

(i) Number and Title indicating the serial number of the table and subject mater of the table.
(ii) stub i.e. the column indicating the headings or rows.
(iii) Caption i.e. the headings of the column (other than stub)
(iv) Body i.e. figures to be entered in the table
(v) Foot-note is source from which the data have been obtained.

Thus table should be arranged as follows :-

Table

Title

Stub	**Caption**	**Total**
	Body	
Total		

Footnote :-

Types of Tabulation :

Mainly there are two types of tables – Simple and Complex. Simple tabulation reveals information regarding one or more groups of independent question, while complex table gives information about one or more interrelated questions.

4.2 FREQUENCY DISTRIBUTION

Frequency of a value of a variable is the number of times it occurs in a given series of observations. A tally-sheet may be used to calculate the frequencies from the raw data (primary data not arranged in the Tabular form). A tally-mark (/) is put against the value when it occurs in the raw data. The following example shows how raw-data can be represented by a tally-sheet :

Example 2 : Raw data Marks in Mathematics of 50 students.

37	47	32	26
21	41	38	41
50	45	52	46
37	45	31	40
44	48	46	16
30	40	36	32
47	37	47	50
40	45	51	52
38	26	41	33
38	39	37	32
40	38	50	38
48	41	36	41
41	52		

Table

Tally-sheet of the given raw data

Marks (x)	Tally-Marks	Frequencies (f)	Marks (x)	Tally-Marks	Frequencies (f)
16	/	1	40	////	4
21	/	1	41	~~////~~	5
26	//	2	43	/	1
30	/	1	44	/	1
31	/	1	45	///	3
32	///	3	46	//	2
33	/	1	47	///	3
36	//	2	48	//	2
37	////	4	50	////	4
38	~~////~~	5	51	/	1
39	/	1	52	//	2
Total		22	Total		28
			Total Frequency		50

Such a representation of the data is known as the Frequency Distribution.

The number of classes should neither be too large nor too small. It should not exceed 20 but should not be less than 5, normally, depending on the number of observations in the raw data.

4.2.1. Group Frequency Distribution :

When large masses of raw data are to be summarised and the identity of the individual observation or the order in which observations arise are not relevant for the analysis, we distribute the data into classes or categories and determine the number of individuals belonging to each class, called the class-frequency.

A tabular arrangement of raw data by classes where the corresponding class-frequencies are indicated is known as Grouped Frequency distribution.

Grouped Frequency Distribution of Marks of 50 students in Mathematics

Serial No.	Marks	No. of Students
1	16-20	1
2	21-25	1
3	26-30	3
4	31-35	5
5	36-40	16
6	41-45	10
7	46-50	11
8	51-55	3
Total		50

4.2.2. Few Terms (associated with grouped frequency distribution) :

(a) Class-interval

(b) Class-frequency, total frequency

(c) Class-limits (upper and lower)

(d) Class boundaries (upper and lower)

(e) Mid-value of class interval (or class mark)

(f) Width of class interval

(g) Frequency denisty

(h) Percentage Frequency.

(a) ***Class-interval*** **:** In the above table, class intervals are 16-20, 21-25 …. etc. In all there are eight class-intervals.

If, however, one end of class-interval is not given then it is known as open-end class. For example, less than 10, 10-20, 20-30, 30 and above. The class-interval having zero frequency is know as empty class.

(b) ***Class frequency*** **:** The number of observations (frequency) in a particular class-interval is known as class-frequency. In the table, for the class-interval 26-30, class frequency is 3 and so on. The sum of all frequencies is total frequency. Here in the table total frequency is 50.

(c) ***Class limits*** **:** The two ends of a class-interval are called class-limits.

(d) ***Class boundaries*** **:** The class boundaries may be obtained from the class limits as follows :

Lower class-boundary = lower class limit – ½ d

Upper class-boundary = upper class limit + ½ d

Where d = common difference between upper class of any class-interval with the lower class of the next class-interval. In the table d = 1.

Lower class boundary $= 16 - \frac{1}{2} \times 1 = 16 - 0.5 = 15.5$

Upper class boundary $= 20 + \frac{1}{2} \times 1 = 20 + 0.5 = 20.5$

Again, for the next class-interval, lower class-boundary = 20.5, upper class boundary = 25.5 and so on.

(e) ***Mid value*** **:** (or class mark). It is calculated by adding the two class limits divided by 2.

In the above table : for the first class-interval

Mid-value $= \frac{16 + 20}{2} = \frac{36}{2} = 18$

For the next one, mid value $= \frac{21 + 25}{2} = 23$ and so on.

(f) ***Width*** **:** The width (or size) of a class interval is the difference between the class-boundaries (not class limits)

Width = Upper class boundary – lower class boundary

For the first class, width = 20.5 – 15.5 = 5

For the second class width = 25.5 – 20.5 = 5, so on.

(g) ***Frequency density*** : It is the ratio of the class frequency to the width of that

class-interval i.e. frequency density = $\frac{\text{class frequency}}{\text{width of the class}}$

For the first class frequency density = $\frac{1}{5} = 0.2$

For the third class frequency density = $\frac{3}{5} = 0.6$

(h) ***Percentage frequency* :** It is the ratio of class-frequency to total frequency expressed as percentage.

i.e. percentage frequency = $\frac{\text{class frequency}}{\text{width of the class}} \times 100$

In the table for the frequency 5, % frequency = $\frac{5}{50} \times 100 = 10$ and so on.

4.2.3. *Cumulative Frequency distribution* :

As the name suggests, in this distribution, the frequencies are cumulated. This is prepared from a grouped frequency distribution showing the class boundaries by adding each frequency to the total of the previous one, or those following it. The former is termed as Cumulative frequency of less than type and the latter, the cumulative frequency of greater than type.

Example 3 : The following is an array of 65 marks obtained by students in a certain examination : –

26	45	27	50	45
32	36	41	31	41
48	27	46	47	31
34	42	45	31	28
27	49	48	47	32
33	35	37	47	28
46	26	46	31	35
33	42	31	41	45
42	44	41	36	37
39	51	54	53	38
55	39	52	38	54
36	37	38	56	59
61	65	64	72	64

Draw up a frequency distribution table classified on the basis of marks with class-intervals of 5.

Class-intervals Of marks	Tally marks	Frequency
25-29	~~||||~~ //	7
30-34	~~||||~~ ~~||||~~	10
35-39	~~||||~~ ~~||||~~ ///	13
40-44	~~||||~~ ///	8
45-49	~~||||~~ ~~||||~~ ///	13
50-54	~~||||~~ /	6
55-59	///	3
60-64	///	3
65-69	/	1
70-74	/	1
Total		65

Now the required frequency distribution is shown below :

Frequency distribution of marks obtained by 65 students

Marks	Frequency
25-29	7
30-34	10
35-39	13
40-44	8
45-49	13
50-54	6
55-59	3
60-64	3
65-69	1
70-74	1
Total	65

4.3 GRAPHICAL REPRESENTATION OF FREQUENCY DISTRIBUTION

4.3.1. HISTOGRAM (when C.I. are equal)

Frequency distribution can be presented in the graphical form. Such graphs are easily perceived by the mind and gives a birds eye view and they are more appealing than the tabulated data. The graph also helps in comparative study of two or more frequency distributions with regards to their shapes and patterns. The most commonly used graphs for charting a frequency distribution are as follows –

Histogram

This graphical method is most widely used in practice. Histogram is a series of adjacent vertical bars whose height is equal to the frequencies of the respective classes & width is equal to the class interval.

Construction of Histogram – While constructing Histogram the variable taken on X-axis & frequency of Y-axis.

(i) Histogram with equal classes – When class intervals are equal, take the frequency on Y-axis and the variable on X-axis and adjacent rectangles are constructed. The height of these rectangles would be exactly equal (or proportional) to the frequency of the given class.

(ii) Histogram with unequal classes – If the classes are not uniform, then frequencies have to be adjusted by the adjustment factor.

First find the class having the lowest class interval. This is taken as the starting point.

$$\text{Adjustment factor} = \frac{\text{Width / magnitude of the class}}{\text{Lowest width of class in the series}}$$

$$\text{Adjusted frequency of a class} = \frac{\text{Given frequency}}{\text{Adjustment factor}}$$

For e.g. the class interval in 70 – 90 its width is 20. If the lowest width in the series is 5, then the adjustment factor is 20/5 = 4 & the corresponding frequency would be divided by 4 to get the required adjusted frequency needed for the graph.

If only midpoints are given then upper & lower limits of various classes have to be calculated & then only histogram would be constructed.

Frequency Polygon

Frequency polygon could be drawn by first drawing histogram & then joining all the midpoints of the tops (upper side) of the adjacent rectangle of the histogram by straight line graphs. The figure so obtained is called a frequency polygon. It should be noted that it is necessary to close the polygon at both ends by extending them to the base line so that it meets the X-axis at the mid points of the two hypothetical classes i.e. the class before the first class & the class after the last class having the zero frequency.

Frequency polygon could alternatively be drawn without first drawing the histogram. This could be done by plotting the frequencies of different classes (along Y-axis) against the mid values of corresponding classes (along X axis). These points are joined by straight line to get a frequency polygon. Here also this polygon would be closed at both ends by extending them to meet the X-axis.

OGIVE

Ogive of cumulative frequency polygon: If the cumulative frequencies are plotted against the class-boundaries and successive points are joined by straight lines, we get what is known as Ogive (or cumulative frequency polygon). There are two types of Ogive.

(a) Less than type – Cumulative Frequency from below are plotted against the upper class-boundaries.

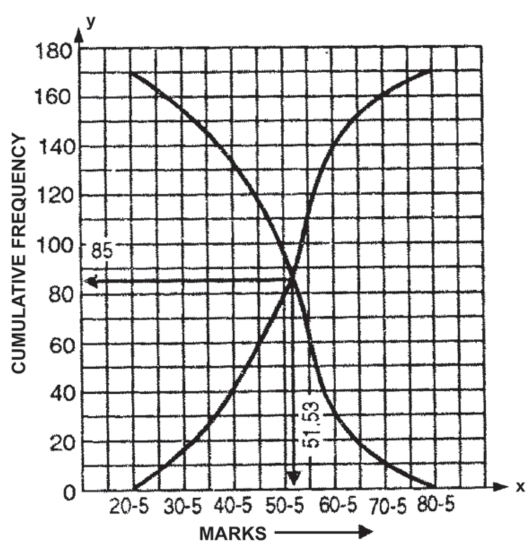
(b) Greater than type – Cumulative frequencies from above are plotted corresponding lower boundaries.

The former is known as less than type, because the ordinate of any point on the curve (obtained) indicates the frequency of all values less than or equal to the corresponding value of the variable reprpresented by the abscissa of the point.

Similarly, the latter one is known as the greater than type.

Frequency Distribution of marks obtained by 170 students.

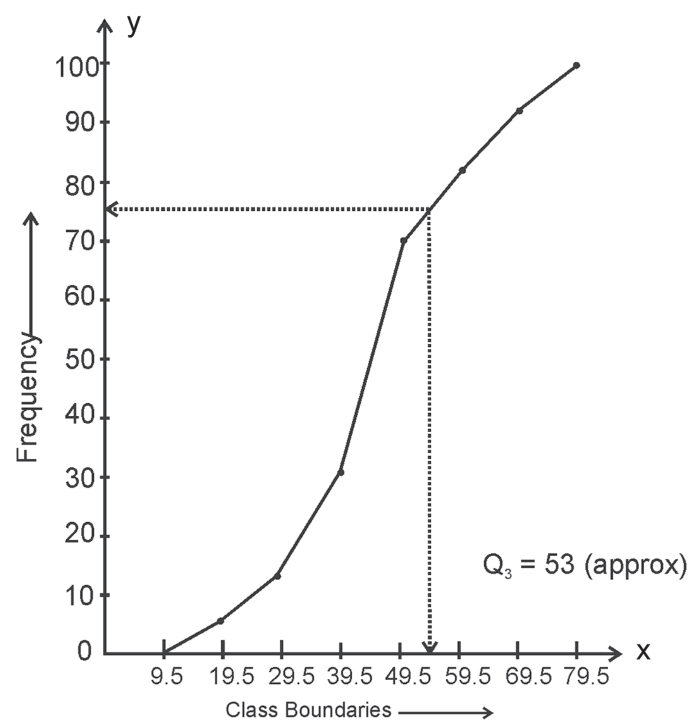
Class-intervals	Class	Frequency	Cumulative Frequency	
			from below (less than 30.5, 40.5 etc	from above (greater than 20.5, 30.5 etc)
21-30	20.5-30.5	15	15	170
31-40	30.5-40.5	25	40	155
41-50	40.5-50.5	40	80	130
51-60	50.5-60.5	60	140	90
61-70	60.5-70.5	20	160	30
71-80	70.5-80.5	10	170	10
Total		170	–	–



Ogives

Note. From the above figure, it is noted that the ogives cut at a point whose ordinate is 85, i.e. half the total frequency corresponding and the abscissa is 51.33 which is the median of the above frequency distribution (see the sum on median in the chapter of Average). Even if one ogive is drawn, the median can be determined by locating the abscissa of the point on the curve, whose cumulative frequency is N/2. Similarly, the abscissa of the points on the *less than type* corresponding to the cumulative frequencies N/4 and 3N/4 give the Q_1 (first quartile) and Q_3 (third quartile) respectively, (Q1, Q3 will be discussed after median in the chapter of Average).

Ogive (less than type) of given data :



Less than ogive

From the above graph we find that Q_3 = 53 (approx).

Circular Diagram (or Pie diagram) : It is a pictorial diagram in the form of circles where whole area represents the aggregate and different sectors of the circle, when divided into several parts, represent the different components.

For drawing a circular diagram, different components are first expressed as percentage of the whole. Now since 100% of the centre of a circle is 360 degrees. 1% corresponds to 3.6 degrees. If p be the percentage of a certain component to the aggregate, then (p × 3.6) degrees will be the angle, which the corresponding sector subtends at the centre.

Note : A pie diagram is drawn with the help of a compass and a diagonal scale or a protractor. Different sectors of the circle representing different components are to be marked by different shades or signs.

Example 10: The expenditure during Second Five-year Plan in West Bengal is shown as below :

	(₹ in Crores)
On Industries	127.00
" Irrigation	92.50
" Agriculture	100.00
" Transports & Roads	92.50
" Miscellaneous	68.00
	480.00

– To represent the data by circular diagram.

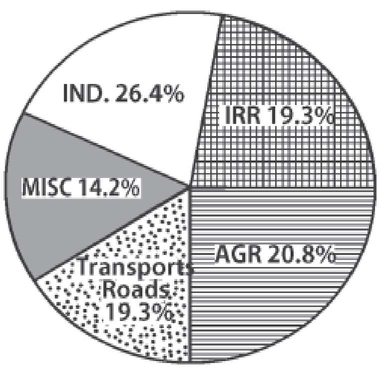
First we express each item as percentage of the aggregate.

Industries	$= \frac{127.00}{480.00} \times 100 = 26.4$
Irrigation	= 19.3
Agriculture	= 20.8
Transports & Roads	= 19.3
Miscellaneous	= 14.2

Now 1% corresponds to 3.6 degrees. So the angles at the centre of the corresponding sectros are (in degrees) :

Industries	= 26.4 × 3.6 = 95.0
Irrigation	= 19.3 × 3.6 = 69.5
Agriculture	= 20.8 × 3.6 = 74.9
Transp. & Roads	= 19.3 × 3.6 = 69.5
Miscellaneous	= 14.2 × 3.6 = 51.1

Now with the help of compass and protractor (or diagonal scale) the diagram is drawn.



Pie Chart

Note: Additions of all percentages of the items should be equal to 100 and also the addition of all the angles should be equal to 360° (approx).

It two aggregates with their components are to be compared, then two circles are required to be drawn having areas proportionate to the ratio of the two aggregates.

Study Note - 5

MEASURES OF CENTRAL TENDENCY AND MEASURES OF DISPERSION

This Study Note includes

5.1 MEASURES OF CENTRAL TENDENCY OR AVERAGE

INTRODUCTION :

A given raw statistical data can be condensed to a large extent by the methods of classification and tabulation. But this is not enough. For interpreting a given data we are to depend on some mathematical measures. Such a type of measure is the measure of Central Tendency.

By the term of 'Central Tendency of a given statistical data' we mean that central value of the data about which the observations are concentrated . A central value which 'enables us to comprehend in a single effort the significance of the whole is known as Statistical Average or simply **average.**

The three common measures of Central Tendency are :

(i) Mean

(ii) Median

(iii) Mode

The most common and useful measure is the mean. As we proceed, we shall discuss the methods of computation of the various measures.

In all such discussions, we need some very useful notations, which we propose to explain before proceeding any further.

(i) Index or Subscript Notation :

Let X be a variable assuming n values $x_1, x_2, \ldots x_3$, We use the symbol x_j (read "x sub j") to denote any of the above mentioned n numbers. The letter j, which can stand for any of the numbers $x_1, x_2, \ldots x_n$ is called a subscript notation of index. Obviously, any letter other than j, as I, k, p, q and s could be used.'

(ii) Summation Notation :

The symbol $\sum_{j=1}^{n} X_j$ is used to denote the sum x_j's from j = 1 to j = n. By definition.

$$\sum_{j=1}^{n} X_j = x_1 + x_2 + \ldots + x_n$$

Example 1 : $\sum_{j=1}^{n} X_j Y_j = X_1 Y_1 + X_2 Y_2 + \ldots + X_n Y_n$

Some important result :

(i) $\sum_{j=1}^{n}(x_j + y_j) = \sum_{j=1}^{n} x_j + \sum_{j=1}^{n} y_1$

(ii) $\sum_{j=1}^{n} A = \frac{A + A + ... + A}{[n \text{ times}]} = nA$ (A is constant)

(iii) $\sum_{j=1}^{n} Ax_j = Ax_1 + Ax_2 + ... + Ax_n$

5.1.1. MEAN :

There are three types of mean :

(i) Arithmetic Mean (A.M.) (ii) Geometric Mean (G. M.) (iii) Harmonic Mean (H.M.)

Of these the Arithmetic mean is the most commonly used. In fact, if not specifically mentioned by mean we shall always refer to arithmetic Mean (AM) and calculate accordingly.

1. Arithmetic Mean :

(i) Simple Arithmetic mean : (Calculating mean from ungrouped data)

The simple arithmetic mean $(\bar{x})$ of a given series of values, say, $x_1, x_2, ... x_n$ is defined as the sum of these values divided by their total number : thus

$$\bar{x}\,(\text{xbar}) = \frac{x_1 + x_2 + ... + x_n}{n} = \frac{\sum_{j=1}^{n} x_j}{n} = \frac{\sum x}{n}$$

Note. Often we do not write x_j , x means summation over all the observations.

Example 1 : Find the arithmetic mean of 3,6,24 and 48.

Required A.M.= $\frac{3+6+24+48}{4} = \frac{81}{4} = 20.25$

(ii) Weighted Arithmetic Mean : (Calculating the mean from grouped data)

If the number $x_1, x_2, ... x_n$ occur $f_1, f_2.......f_n$ times respectively (i.e. occur with frequencies $f_1, f_2 ....f_n$) the arithmetic mean is

$$\bar{x} = \frac{\sum_{j=1}^{n} x_j f_j}{\sum_{j=1}^{n} f_j} = \frac{f_1x_1 + f_2x_2..... + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum fx}{\sum f} = \frac{\sum fx}{N}$$

Where $N = \sum f$ is the total frequency, i.e., total number of cases. This mean $\bar{x}$ is called the weighted Arithmetic mean, with weights $f_1, f_2 ....f_n$ respectively.

In particular, when the weights (or frequencies) $f_1, f_2......f_n$ are all equal. We get the simple Arithmetic Mean.

Example 2 : If 5, 8, 6 and 2 occur with frequency 3, 2, 4 and 1 respectively, find the Arithmetic mean.

Arithmetic mean $= \frac{(3\times5)+(2\times8)+(4\times6)+(1\times2)}{3+2+4+1} = \frac{15+16+24+2}{10} = \frac{57}{10} = 5.7 \quad \therefore \quad \overline{x} = 5.7$

Calculation of Arithmetic Mean (or simply Mean) from a grouped frequency distribution — Continuous Series.

(i) Ordinary method (or Direct Method)

In this method the mid-values of the class-intervals are multiplied by the corresponding class-frequencies. The sum of products thus obtained is divided by the total frequency to get the Mean. The mean $\overline{x}$ is given by

$\overline{x} = \frac{(\quad fx)}{N}$, where x = mid-value of a class and N = total frequency

Example 3 : Calculate the mean of daily-wages of the following table :

Wages (₹)	No. of workers
4–6	6
6–8	12
8–10	17
10–12	10
12–14	5

Solution:

Table : Calculation of Mean Daily Wages

Class Interval	Mid values (₹) x	Frequency f	fx
4–6	5	6	30
6–8	7	12	84
8–10	9	17	153
10–12	11	10	110
12–14	13	5	65
Total	––	N = 50	fx = 442

∴ Mean Daily Wages

$= \frac{fx}{N} = \frac{442}{50} = ₹\ 8.84$

(ii) **Shortcut Method (Method of assumed Mean)**

In this method, the mid-value of one class interval (preferably corresponding to the maximum frequency lying near the middle of the distribution) is taken as the assumed mean (or the arbitrary origin) A and the deviation from A are calculated. The mean is given by the formula :

$\bar{x} = A + \frac{fd}{N}$ where, d = x – A = (mid value) – (Assumed Mean).

Step deviation method :

$\bar{x} = A + \frac{fd'}{f} \times i$, where $d' = \frac{x - A}{i}$ i = scale (= width of C.I.)

Example 4: Compute the Arithmetic Mean of the following frequency distribution :

Marks	No. of student
20–29	5
30 –39	11
40– 49	18
50 –59	22
60 –69	16
70–79	8

Solution:

Table: Calculation of Arithmetic Mean

Class Interval	Mid values x	Deviation from 54.5 d = x – 54.5	frequency f	fd
20–29	24.5	– 30	5	– 150
30–39	34.5	– 20	11	– 220
40 –49	44.5	–10	18	–180
50–59	54.5 (=A)	0	22	0
60–69	64.5	10	16	160
70 – 79	74.5	20	8	160
Total	—	—	N = 80	– 550 +320
				fd = – 230

∴ Arithmetic Mean

$= A + \frac{fd}{N} = 54.5 + \frac{-230}{80}$

= 54.5 – 2.875 = 51.625 = 51.6 (approx).

(iii) **Method of Assumed mean (by using step deviations)**

Table : Calculation of Arithmetic Mean

Class	Mid-points	$d' = \frac{x - A}{i}$	f	fd'
19.5–29.5	24.5	– 3	5	– 15
29.5–39.5	34.5	– 2	11	– 22
39.5 –49.5	44.5	–1	18	–18
49.5–59.5	54.5=A	0	22	0
59.5–69.5	64.5	1	16	16
69.5 – 79.5	74.5	2	8	16
Total	—	—	N = 80	$fd' = -23$

$$\text{A.M.} = A + \frac{fd'}{N} \times i = 54.5 - \frac{23}{80} \times 10 = 54.5 - 2.88 = 51.6(\text{approx}).$$

5.1.1.1Calculation of A. M. from grouped frequency distribution with open ends

If in a grouped frequency distribution, the lower limit of the first class or the upper limit of the last class are not known, it is difficult to find the A.M. When the closed classes (other than the first and last class) are of equal widths, we may assume the widths of the open classes equal to the common width of closed class and hence determine the AM. But we can find Median or Mode without assumption.

Properties of Arithmetic Mean :

1. The sum total of the values fx is equal to the product of the number of values of their A.M.

   e.g. $N\bar{x} = \quad fx$.

2. The algebraic sum of the deviations of the values from their AM is zero.

   If $x_1, x_2 \ldots\ldots x_n$ are the n values of the variable x and $\bar{x}$ their AM then $x_1 - \bar{x}, x_2 - \bar{x}, \ldots.. x_n - \bar{x}$ are called the deviation of $x_1, x_2 \ldots\ldots\ldots\ldots x_n$ respectively from from $\bar{x}$

   Algebraic sum of the deviations $= \sum_{j=1}^{n} (x_j - \bar{x})$

   $= (x_1 - \bar{x}) + (x_2 - \bar{x}) + \ldots.. + (x_n - \bar{x}) = (x_1 + x_2 + \ldots. + x_n) - n\bar{x} = n\bar{x} - n\bar{x} = 0$

Similarly, the result for a weighted AM can be deduced.

3. If group of n_1 values has AM. $\bar{x}_1$ and another group of n_2 values has AM $\bar{x}_2$, then A.M. $(\bar{x})$ of the composite group (i.e. the two groups combined) of $n_1 + n_2$ values is given by :

   $\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$ In general, for a group the AM $(\bar{x})$ is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3 + \ldots + n_r\bar{x}_r}{n_1 + n_2 + \ldots + n_r} = \frac{\sum_{j=1}^{r} n_1\bar{x}_j}{\sum_{j=1}^{r} n_j}$$

Example 5: The means of two samples of sizes 50 and 100 respectively are 54.1 and 50.3. Obtain the mean of the sample size 150 obtained by combining the two sample.

Here, $n_1 = 50$, $n_2 = 100$, $\bar{x}_1 = 54.1$, $\bar{x}_2 = 50.3$

$$\therefore \text{Mean } (\bar{x}) = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{50 \times 54.1 + 100 \times 50.3}{50 + 100}$$

$$= \frac{2705 + 5030}{150} = \frac{7735}{150} = 51.57 \text{ (approx.)}$$

5.1.1.2. Finding of missing frequency :

In a frequency distribution if one (or more) frequency be missing (i.e. not known) then we can find the missing frequency provided the average of the distribution is known. The idea will be clear from the following example :

For one missing frequency :

Example 6 : The AM of the following frequency distribution is 67.45. Find the value of f_3,

Let A = 67. Now using the formula.

Height (inch)	Frequency
61	15
64	54
67	f_3
70	81
73	24

Solution:

Calculation of missing frequency

Table : Calculation of Mean

Height (x)	Frequency(f)	d = x – A	d´ = d / 3	fd´
61	15	–6	–2	– 30
64	54	–3	–1	– 54
67	f_3	0	0	0
70	81	3	1	81
73	24	6	2	48
Total	N =174 + f_3	–	–	Σfd′ = 45

Let A = 67. Now using the formula

$\bar{x} = A + \frac{fd'}{f} \times i$, we get $67.45 = 67 + \frac{45}{174 + f_3} \times 3$

or, $0.45 = \frac{135}{174 + f_3}$ or, $78.30 + 0.45 f_3 = 135$

or, $0.45f_3 = 56.70$ or, $f_3 = \frac{56.70}{0.45} = 126$

For two missing frequencies :

Example 7: The A.M. of the following frequency distribution is 1.46

No. of accidents	No. of days
0	46
1	f_1
2	f_2
3	25
4	10
5	5
Total	200

Find f_1 and f_2 ?

Let, x = No. of accidents, f = No. of days

Solution:

Table : Calculation of Mean

x	f	d = x – 2	fd
0	46	–2	–92
1	f_1	–1	$-f_1$
2	f_2	0	0
3	25	1	25
4	10	2	20
5	5	3	15
Total	N = 200	––	$\Sigma fd = -32 - f_1$

$$AM = A + \frac{fd}{f}$$

or, $1.46 = 2 + \frac{-32 - f_1}{200}$ or, $-0.54 = \frac{-(32 + f_1)}{200}$ or, $108 = 32 + f_1$, or, $f_1 = 76$,

$f_2 = 200 - (46 + 76 + 25 + 10 + 5) = 38$

Example 8 : Arithmetic mean of the following frequency distribution is 8.8. Find the missing frequencies :

Wages (₹)	4–6	6–8	8–10	10–12	12–14	Total
No. workers :	6	––	16	––	5	50

Solution:

Table : Calculation of Arithmetic Mean

wages (₹)	x	f	fx
4–6	5	6	30
6–8	7	f_1	$7f_1$
8–10	9	16	96
10–12	11	f_2	$11f_2$
12–14	13	5	65
Total		$N = 27+f_1+f_2$	$191+7f_1+11f_2 = \sum fx$

$\therefore$ $27 + f_1 + f_2 = 50$ or, $f_1 + f_2 = 23$

$$\bar{x} = \frac{\sum fx}{\sum f} \text{ or, } 8.8 = \frac{191+7f_1+11f_2}{27+f_1+f_2}, \quad 8.8 = \frac{191+7f_1+11(23-f_1)}{27+23}$$

or, $8.8 \times 50 = 191 + 253 - 4f_1$ or, $4f_1 = 444 - 440 = 4$ or, $f_1 = 1$ i.e. $f_2 = 23 - 1 = 22$

5.1.1.3. Wrong Observation :

After calculating A.M. $(\bar{x})$ of n observations if it is detected that one or more observations have been taken wrongly (or omitted), then corrected calculation of A.M. will be as follows :

Let wrong observations x_1, y_1 being taken instead of correct values x, y then corrected $\sum x$ = given $\sum x - (x_1 + y_1) + (x + y)$, in this case total no. of observations will be same.

Example 9. The mean of 20 observations is found to be 40. Later on, it was discovered that a marks 53 was misread as 83. Find the correct marks.

Wrong $\sum x = 20 \times 40 = 800$, Correct $\sum x = 800 - 83 + 53 = 770$

$\therefore$ Correct $\bar{x} = \frac{770}{20} = 38.5$

Example 10. A.M. of 5 observations is 6. After calculation it has been noted that observations 4 and 8 have been taken in place of observations 5 and 9 respectively. Find the correct A.M.

$$\bar{x} = \frac{\sum x}{n} \text{ or, } 6 = \frac{\sum x}{5} \text{ or, } \sum x = 30, \qquad \text{corrected } \sum x = 30 - (4+8) + (5+9) = 32$$

Corrected A.M. $= \frac{32}{5} = 6.4$

5.1.1.4. Calculation of A.M. from Cumulative Frequency Distribution

At first we are to change the given cumulative frequency distribution into a general form of frequency distribution, then to apply the usual formula to compute A.M. the idea will be clear from the following examples.

Example 11 : Find A.M. of the following distributions :

	₹	c.f.		Marks	c.f.
(i)	less than 4	2	(ii)	More than 0 and above	10
	less than 8	6		More than 5 and above	8
	less than 12	13		More than 10 and above	5
	less than 16	18		More than 15 and above	1
	less than 20	20		More than 20 and above	0

(i) The difference between any two variables is 4; so the width of class-intervals will be 4. Accordingly, we get the general group frequency distribution as follows :

Solution:

Table : Calculation of A.M.

₹	f	₹	x	f	d = (x–A)	fd
0–4	2	0–4	2	2	–8	–16
4–8	4(=6–2)	4–8	6	4	–4	–16
8–12	7(=13–6)	8–12	10 = (A)	7	0	0
12–16	5 (= 18–13)	12–16	14	5	4	20
16–20	2 (= 20 –18)	16 –20	18	2	8	16
		Total	––	20	––	4

Let A = 10

$$\overline{x} = A + \frac{fd}{f} = 10 + \frac{4}{20} = 10 + 0 \cdot 2 = ₹\,10 \cdot 2$$

Table : Calculation of A.M.

Marks	f	Marks.	x	f	d	d´	fd´
0–5	2 (=10–8)	0–5	2.5	2	–5	–1	–2
5–10	3 (= 8 –5)	5–10	7.5	3	0	0	0
10–15	4 (=5–1)	10 –15	12.5	4	+ 5	1	4
15–20	1(= 1–0)	15–20	17.5	1	+10	2	2
	Total	––	10	––	––	4	

Let A = 7.5

A.M. = $A + \frac{\sum fd'}{\sum f} \times i = 7.5 + \frac{4}{10} \times 5 = 7.5 + 2 = 9.5$ marks.

5.1.1.5. Advantages of Arithmetic Mean

(i) It is easy to calculate and simple to understand.

(ii) For counting mean, all the data are utilised. It can be determined even when only the number of items and their aggregate are known.

(iii) It is capable of further mathematical treatment.

(iv) It provides a good basis to compare two or more frequency distributions.

(v) Mean does not necessitate the arrangement of data.

5.1.1.6. Disadvantages of Arithmetic Mean

(i) It may give considerable weight to extreme items. Mean of 2, 6, 301 is 103 and more of the values is adequately represented by the mean 103.

(ii) In some cases, arithmetic mean may give misleading impressions. For example, average number of patients admitted in a hospital is 10.7 per day, Here mean is a useful information but does not represent the actual item.

(iii) It can hardly be located by inspection.

Example 12 : Fifty students appeared in an examination. The results of passed students are given below :

Marks	No. of students
40	6
50	14
60	7
70	5
80	4
90	4

The average marks for all the students is 52. Find out the average marks of students who failed in the examination.

Table : Calculation of Arithmetic Mean

Marks (x)	f	fx
40	6	240
50	14	700
60	7	420
70	5	350
80	4	320
90	4	360
Total	N = 40	fx = 2390

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{2390}{40} = 59.75,\ n_1 = 40$$

Let average marks of failed students = $\bar{x}_2$, $n_2 = 10$

$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$ or, $52 = \frac{40 \times 59.75 + 10\bar{x}_2}{50}$

or, $\bar{x}_2 = 21$ (on reduction) ∴ required average marks = 21.

Example 13. From the following frequency table, find the value of x if mean is 23.5

Class :	50–59	40–49	30–39	20 –29	10–19	0–9
frequency :	x – 4	x – 2	x + 3	x + 5	x + 10	x –2

Solution:

Table : Calculation of Arithmetic Mean

Class	*mid. pt.* (x)	*f*	*d* = x– 34.5	*d´* (d/i)	*fd´*
50–59	54.5	x – 4	20	2	2x – 8
40 –49	44.5	x –2	10	1	x – 2
30 – 39	34.5	x + 3	0	0	0
20 – 29	24.5	x + 5	–10	– 1	– x –5
10 – 19	14.5	x + 10	–20	– 2	– 2x – 20
0 – 9	4.5	x – 2	– 30	– 3	–3x + 6
Total		N = 6x + 10			fd´ = – 3x –29

∴ Here, i = width of the class = 10.

$\bar{x} = A + \frac{\sum fd'}{\sum f} \times i$, or $23.5 = 34.5 + \frac{-3x - 29}{6x + 10} \times 10$ or, x = 5 (on reduction).

Example 14 : The mean salary of all employees of a company is ₹ 28,500.The mean salaries of male and female employees are ₹ 30,000 and ₹ 25,000 respectively. Find the percentage of males and females employed by the company.

Let number of male employees be n_1 and that of female be n_2. We know $\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

or, $28500 = \frac{30000n_1 + 25000n_2}{n_1 + n_2}$

or, $7n_2 = 3n_1$ (on reduction) or, $\frac{n_1}{n_2} = \frac{7}{3}$

Percentage of n_1 (male) $= \frac{7}{10} \times 100 = 70\%$

,, ,, n_2 (female) $= \frac{3}{10} \times 100 = 30\%$

SELF EXAMINATION QUESTIONS :

1. The weight of 6 persons are as follows (in kg.) 70, 42, 85, 75, 68, 55. Find the mean weight.

   [Ans. 65.83 kg.]

2. Find A.M. of the following numbers :

   (i) 1, 2, 3, …… upto 10th term [Ans. 5.5]

   (ii) The first 10 even numbers [Ans. 11]

   (iii) The first 10 odd numbers [Ans. 10]

3. Find A.M. of the following numbers :

   (i) 77, 73, 75, 70, 72, 76, 75, 71, 74, 78 [Ans. 74.10]

   (ii) 4, 5, 6, 7, 5, 4, 8, 6, 2, 5, 3 [Ans. 5]

4. Find A.M. of the given frequency distribution :

(i)

Weight (kg.)	Persons
50	15
55	20
60	25
65	30
70	30
Total	120

[Ans. 61.67 kg.]

(ii)

Weight (kg.)	Workers
20	8
21	10
22	11
23	16
24	20
25	25
26	15
27	9
28	6

[Ans. 24.05 Kg.]

5. Find the weekly average wage from the given frequency :

Wages (₹)	No. of Workers
30–40	80
40–50	20
50–60	40
60–70	18
70–80	10
80–90	4

[Ans. ₹ 47.44]

6. Find A.M. from the following table :

Wages (₹)	No. of Workers
20–25	200
25–30	700
30–35	900
35–40	800
40–45	600
45–50	400

[Ans. ₹ 35.97]

7. Compute A.M. of the following distribution :

Class Interval	*Frequency*
1–4	6
4–9	12
9–16	26
16–27	20

[Ans. 8.68]

8. A.M. of the following distribution is 124 lb.

Weight (lb)	*No. of Persons*
100	1
110	2
120	3
135	2
x + 5	2
Total	10

[Ans. 140]

Find the value of x.

[Hint. Use direct method

9. A.M. of the following frequency distribution is 5.6. Find the missing frequency.

x	f
2	4
4	2
6	—
8	3
10	2

[Ans. 4]

10. A.M. of the distribution is ₹ 56.46. Find missing frequencies.

Daily Wages (₹)	*Frequency*
45	5
50	48
55	f_3
60	30
65	f_5
70	8
75	6
Total	150

[Ans. 41, 12]

5.1.2. GEOMETRIC MEAN (G. M.)

Definition. : The geometric mean (G) of the n positive values x_1, x_2, x_3 ....x_n is the n^{th} root of the product of the values i.e. $G = \sqrt[n]{x_1.x_2.....,x_n}$ It means, $G = (x_1. x_2,........ x_n)^{1/n}$

Now taking logarithms on both sides, we find

$$\log G = \frac{1}{n}\log(x_1.x_2........,x_n) = \frac{1}{n}(\log x_1 + ... + \log x_n) = \frac{1}{n}\sum \log x .....(1)$$

$$\therefore G = \text{antilog}\left[\frac{1}{n}\sum \log x\right]$$

Thus, from formula (1) we find that the logarithm of the G. M. of x_1, x_2 ....., x_n = A.M. of logarithms of x_1, x_2, ..., x_n.

Properties :

1. The product of n values of a variate is equal to the n^{th} power of their G. M. i.e., x_1, x_2, ..., $x_n = G^n$ (it is clear from the definition)]
2. The logarithm of G. M. of n observations is equal to the A.M. of logarithms of n observations. [Formula (1) states it]

3. The product of the ratios of each of the n observations to G. M. is always unity. Taking G as geometric mean of n observations $x_1, x_2, \ldots\ldots, x_n$ the ratios of each observation to the geometric mean are $\frac{x_1}{G}, \frac{x_2}{G} \ldots\ldots \frac{x_n}{G}$

By definition, $G = \sqrt[n]{x_1, x_2, \ldots\ldots, x_n}$ or, $G^n = (x_1, x_2, \ldots\ldots, x_n)$. Now the product of the ratios.

$$\frac{x_1}{G} \cdot \frac{x_2}{G} \ldots\ldots \frac{x_n}{G} = \frac{x_1.x_2 \ldots . x_n}{G.G.....\text{to n times}} = \frac{G^n}{G^n} = 1$$

4. If G_1, G_2......, are the geometric means of different groups having observations n_1, n_2.........respectively, then the G. M. (G) of composite group is given by

$$G = \sqrt[N]{G_1^{n_1}.G_2^{n_2}....} \text{ where } N = n_1 + n_2 + ...\text{i.e., } \log G = \frac{1}{N}\, n_1 \log G_1 + n_2 \log G_2 + ...$$

Example 15 : Find the G. M. of the number 4, 12, 18, 26.

Solution : $G = \sqrt[4]{4.12.18.26}$; here n = 4

Taking logarithm of both sides,

$$\text{Log } G = \frac{1}{4}(\log 4 + \log 12 + \log 18 + \log 26)$$

$$= \frac{1}{4}(0.6021 + 1.0792 + 1.2553 + 1.4150)$$

$$= \frac{1}{4}(4.3516) = 1.0879$$

$\therefore$ G = antilog 1.0879 = 12.25.

5.1.2.1. Weighted Geometric Mean :

If $f_1, f_2, f_3 \ldots\ldots f_n$ are the respective frequencies of n variates $x_1, x_2, x_3, \ldots\ldots. x_n$, then the weighted G. M. will be

$$G = \left(x_1^{f_1} \times x_2^{f_2} \times x_3^{f_3} \times ..... \times x_n^{f_n}\right)^{1/n} \text{ where } N = f_1 + f_2 + \ldots\ldots + f_n = \sum f$$

Now taking logarithm.

$$\text{Log } G = \frac{1}{N}\,(f_1 \log x_1 + f_2 \log x_2 + f_3 \log x_3 + ... + f_n \log x_n)$$

$$= \frac{1}{N}\sum f \log x. \qquad G = \text{anti log}\left(\frac{1}{N}\sum f \log x\right)$$

Steps to calculate G. M.

1. Take logarithm of all the values of variate x.
2. Multiply the values obtained by corresponding frequency.

3. Find f log x and divide it by f , i.e., calculate $\frac{f \log x}{f}$.

4. Now antilog of the quotient thus obtained is the required G. M. The idea given above will be clear from the following example.

Example 16 : Find (weighted) G. M. of the table given below : —

x	f
4	2
12	4
18	3
26	1

Solution :

Table : Calculation of G.M

x	*f*	*log x*	*f log x*
4	2	0.6021	1.2032
12	4	1.0792	4.3168
18	3	1.2553	3.7659
26	1	1.2553	1.4150
Total	10	–––	10.7019

$$\therefore \log G = \frac{f \log x}{f} = \frac{10.7019}{10} = 1.07019$$

$\therefore$ G = antilog 1.07019 = 11.75

Advantages Geometric Mean

(i) It is not influenced by the extreme items to the same extent as mean.

(ii) It is rigidly defined and its value is a precise figure.

(iii) It is based on all observations and capable of further algebraic treatment.

(iv) It is useful in calculating index numbers.

Disadvantages of Geometric Mean :

(i) It is neither easy to calculate nor it is simple to understand.

(ii) If any value of a set of observations is zero, the geometric mean would be zero, and it cannot be determined.

(iii) If any value is negative, G. M. becomes imaginary.

[Use. It is used to find average of rates of changes.]

SELF EXAMINATION QUESTIONS

1. Find G.M of the following numbers :

(i) 3, 9, 27 [Ans.9]

(ii) 3, 6, 24, 48 [Ans. 12]

2. Weekly wages of 6 workers are 70, 42, 85, 75, 68, 53 (in ₹).

Find the G. M. [Ans. 64.209]

3. Calculate G. M. (upto 2 decimal places) :

(i) 90, 25, 81, $\frac{125}{3}$ [Ans. 69.08]

(ii) 125, $\frac{700}{3}$, 450, 87 [Ans. 183.90]

4. Compute G. M. :

(i) 4, 16, 64, 256 [Ans. 32]

(ii) 1, 2, 4, 8, 16 [Ans. 4]

(iii) 2, 79; 0.375, 1000 [Ans. 10.877]

5. Monthly expenditure of 5 students are as follows :

₹ 125, 130, 75, 10, 45, find G.M. [Ans. ₹ 55.97]

6. Calculate G.M. of 2574, 475, 75, 5, 0.8, 0.005,, 0.0009. [Ans. 2.882]

7. Find G.M. of the table given :

x	f
44.5	2
7.05	3
91.72	4

[Ans. 71.38]

8. Find G.M. of 111, 171, 191, 212, having weight by 3, 2, 4 and 5 respectively. [Ans. 173.4]

9. Increase of productions for the first three years are respectively 3%, 4%and 5%. Find average production of the three years. [Hint : Use G.M] [Ans. 3.9%]

5.1.3. HARMONIC MEAN (H. M.) :

Definition.

The Harmonic Mean (H) for n observations, x_1, x_2,.......x_n is the total number divided by the sum of the reciprocals * of the numbers.

$$\text{i.e. } H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}} = \frac{n}{\cdot \frac{1}{x}}$$

$$\text{Again, } \frac{1}{H} = \frac{\sum \cdot \frac{1}{x}}{n}$$ (i.e. reciprocal of H. M = A. M. of reciprocals of the numbers).

* For ab = 1. $a = \frac{1}{b}$, i.e. a reciprocal of b. And for $\frac{1}{a}$ = b, b is reciprocal of a. Reciprocal of 2 is. $\frac{1}{2}$.

Example 17 : Find the H. M. of 3, 6, 12 and 15.

$$\text{H.M.} = \frac{4}{\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\frac{1}{15}} = \frac{4}{\frac{20+10+5+4}{60}} = \frac{240}{39} = 6.15$$

Example 18 : Find the H.M. of $1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots \frac{1}{n}$

$$H.M. = \frac{n}{1+2+3+\ldots+n} = \frac{n}{\frac{n}{2}(2+n-1)} = \frac{2n}{n(n+1)}$$

[Note. The denominator is in A..P. use $S = \frac{n}{2}\{2a+(n-1)d\}$

Example 19 : A motor car covered distance of 50 miles four times. The first time at 50 m. p. h, the second at 20 m. p. h., the third at 40 m. p. h, and the fourth at 25 m.p.h Calculate the average speed and explain the choice of the average.

$$\text{Average Speed (H.M)} = \frac{4}{\frac{1}{50}+\frac{1}{20}+\frac{1}{40}+\frac{1}{25}} = \frac{4}{\frac{20+50+25+40}{1000}} = 4\times\frac{1000}{135} = 29.63$$

$$= 30 \text{ (app.) m. p. h.}$$

For the statement x units per hour, when the different values of x (i.e. distances) are given, to find average, use H.M. If again hours (i.e., time of journey) are given, to find average, we are to use A.M. In the above example, miles (distances) are given, so we have used H.M.

Weighted H.M. The formula to be used is as follows :

$$\text{H.M.} = \frac{N}{\frac{f_1}{x_1}+\frac{f_2}{x_2}+\ldots..+\frac{f_n}{x_n}}, \quad \sum f = N$$

Example 20 :

(a) A person travelled 20 k.m. at 5 k.m.p.h. and again 24 k.m. at 4 k.m.p.; to find average speed.

(b) A person travelled 20 hours at 5 k.m.p.h. and again 24 hours at 4.m.p.h.; to find average speed.

(a) We are to apply H.M. (weight) in this case, since, distances are given.

$$\text{Average speed (H.M.)} = \frac{20+24}{\frac{20}{5}+\frac{24}{4}} = \frac{44}{4+6} = \frac{44}{10} = 4.4 \text{ k.m.p.h.}$$

(b) We are to apply A.M. (weighted), since times of journey are given.

$$\text{Average speed (A.M.)} = \frac{20\times5+24\times4}{20+24} = \frac{100+96}{44} = \frac{196}{44} = 4.45 \text{ k.m.p.h (app.)}$$

Example 21 : Find the harmonic mean of the following numbers :

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

$$H.M. = \frac{4}{\frac{1}{1}+\frac{1}{1/2}+\frac{1}{1/3}+\frac{1}{1/4}} = \frac{4}{1+2+3+4} = \frac{4}{10} = \frac{2}{5}$$

Example 22 : An aeroplane flies around a square and sides of which measure 100 kms. Each. The aeroplane cover at a speed of 10 Kms per hour the first side, at200 kms per hour the second side, at 300 kms per hour the third side and at 400 kms per hour the fourth side. Use the correct mean to find the average speed round the square.

Here H.M. is the appropriate mean.

Let the required average speed be H kms per hours

$$\text{then } H = \frac{4}{\frac{1}{100}+\frac{1}{200}+\frac{1}{300}+\frac{1}{400}} = \frac{4}{\frac{12+6+4+3}{1200}} = \frac{4\times1200}{25} = 4 \times 48 = 192 \text{ kms/hr.}$$

ADVANTANGES OF HARMONIC MEAN :

(i) Like A.M. and G. M. it is also based on all observations.

(ii) Capable of further algebraic treatment.

(iii) It is extremely useful while averaging certain types of rates and rations.

DISADVANTAGES OF HARMONIC MEAN :

(i) It is not readily understood nor can it be calculated with ease.

(ii) It is usually a value which may not be a member of the given set of numbers.

(iii) It cannot be calculated when there are both negative and positive values in a series or one of more values in zero.

It is useful in averaging speed, if the distance travelled is equal. When it is used to give target weight to smallest item, this average is used.

5.1.4. Relations among A.M., G.M. and H.M. :

1. The Arithmetic Mean is never less than the Geometric Mean, again Geometric Mean is never less than the Harmonic Mean.

i.e. A.M. ≥ G. M. ≥ H. M.

Uses of H.M. : Harmonic mean is useful in finding averages involving rate, time, price and ratio.

Example 23 : For the numbers 2, 4, 6, 8, 10, find GM & HM and show that AM > GM > HM.

$$G. M. = \sqrt[5]{2\cdot4\cdot6\cdot8\cdot10} = (2\cdot4\cdot6\cdot8\cdot10)^{1/5}$$

$$\text{Log GM} = \frac{1}{5}\,(\log 2 + \log 4 + \log 6 + \log 8 + \log 10)$$

$= \frac{1}{5}(0.3010 + 0.6021 + 0.7782 + 0.9031 + 1.0000) = \frac{1}{5} \times 3.5844 = 0.7169$

$\therefore$ G. M. = antilog 0.7169 = 5.211

$$\text{H.M.} = \frac{5}{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}} = \frac{5}{\frac{1}{120}(60+30+20+25+12)}$$

$= 5 \times \frac{120}{137} = \frac{600}{137} = 4.379$

Again A.M. $= \frac{1}{5}(2+4+6+8+10) = \frac{1}{5} \times 30 = 6$

We get A.M. =6, G. M. = 5.211, H.M. = 4.379 i.e. A.M. ≥ G. M. ≥ H. M

Note : In only one case the above relation is not true. When all the variates are equal, we will find that AM = GM = HM

Example 24 : A.M. and G.M. of two observations are respectively 30 and 18. Find the observations. Also find H.M.

Now $\frac{x+y}{2} = 30$ or, x + y = 60 ....(1) again $\sqrt{xy} = 18$

Or, xy = 324 or, (60 – y). y = 324, from (1)

Or, $y^2 - 60y + 324 = 0$ or, (y –54) (y –6) = 0, y = 54, 6

$\therefore$ y = 54 , x = 6 or, y = 6, x = 54.

$\therefore$ Required observations are 6, 54.

$\therefore$ H. M. $= \frac{2}{1/6 + 1/54} = \frac{2}{\frac{9+1}{54}} = 2 \times \frac{54}{10} = 10.80.$

SELF EXAMINATION QESTIONS :

1. Find H.M. of the numbers :

(i) 3, 6, 24, 48 [Ans. 7.1]

(ii) 2, 4, 6, 8 [Ans. 3.84]

2. Calculate H.M. of the following numbers

(i) $1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{10}$ [Ans. 0.18]

(ii) $1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots \frac{1}{n}$ [Ans. $\frac{2}{n+1}$]

(iii) $1, \frac{1}{3}, \frac{1}{5}, \ldots\ldots, \frac{1}{2n-1}$ [Ans. $\frac{1}{n}$]

3. Places A, B and C are equidistant from each other. A person walks from A to B at 5 km.p.h.; from B to C at 5 km.p.h. and from C to A at 4 km.p.h.. Determine his average speed for the entire trip.

[Ans. $4\frac{8}{13}$km.p.h.]

4. A person covered a distance from X to Y at 20 km.p.h.. His average speed is 22 km.p.h.. Is the statement correct? [Hints. Use H.M.] [Ans No.]

5.1.5. MEDIAN :

Definition

If a set of observation are arranged in order of magnitude (ascending or descending), then the middle most or central value gives the median. Median divides the observations into two equal parts, in such a way that the number of observations smaller than median is equal to the number greater than it. It is not affected by extremely large or small observation. Median is, thus an average of position. In certain sense, it is the real measure of central tendency.

So **Median** is the middlemost value of all the observations when they are arranged in ascending order of magnitudes.

5.1.5.1. Calculation of Median :

(A) For simple data or Series of Individual Observations :

Individual observations are those observations (or variates) having no frequencies or frequency is unit every case.

At first, the numbers are to arranged in order of magnitude (ascending or descending). Now for n (the total number of items) odd.

Median = value of $\frac{n+1}{2}$ th item

and for n even

Median = average value $\frac{n}{2}$ th item and $\left(\frac{n}{2}+1\right)$ th item.

or, median = value of $\frac{n+1}{2}$th item (n = odd or even)

[*Note :* $\frac{n+1}{2}$th item gives the location of median, but not its magnitude]

Steps to calculate Median

1. Arrange the data in ascending or descending order. 2. Find n (odd or even). 3. Apply usual formula and calculate.

Example 25 : To find the median of the following marks obtained by 7 students : 4, 12, 7, 9, 14, 17, 16.

(i) Arrangement of marks : 4, 7, 9, 12, 14, 16, 17.

(ii) n = 7 = an odd number

(iii) Median = value of $\frac{n+1}{2}$th item = value of $\frac{7+1}{2}$th item = value of 4 th item = 12 (from the arranged data

∴ median is 12 marks.

[*Note :* Unit of the result will be same as given in original variate.]

Example 26 : To find the median of marks : 4, 12, 7, 9, 14, 17, 16, 21

(i) Arrangement : 4, 7, 9, 12, 14, 16, 17, 21. (ii) n = 8 = an even number.

(iii) Median = average value of $\frac{n}{2}$th item and $\left(\frac{n}{2}+1\right)$th i.e.

= average value of $\frac{8}{2}$th item and the next item

= average value of 4th item and the 5th item

= average value of 12 and 14 marks = $\frac{12+14}{2}$ = 13 marks.

Alternative way

Median = value of $\frac{n+1}{2}$th item = value of $\frac{8+1}{2}$th item = value 4.5th item = $\frac{1}{2}$ (value of 4th item and value of 5th item) = $\frac{1}{2}$(12+14) = $\frac{1}{2}$ × 26 = 13 marks.

(B) For Direct Series (or simple Frequency Distribution)

Cumulative frequency (less than type) is calculated. Now the value of the variable corresponding to the cumulative frequency $\frac{n+1}{2}$ gives the median, when N is the total frequency.

Example 27 : To find the median of the following

x :	1	2	3	4	5	6
y :	7	12	17	19	21	24

Solution :

Table : Calculation of Median

x	f	cum . freq. (c.f)
1	7	7
2	12	19
3	17	36
4	19	55
5	21	76
6	24	100 (= N)
	N = 100	

Now, median = value of $\frac{n+1}{2}$th item = value of $\frac{100+1}{2}$th item = value of 50.5th item.

From the last column, it is found 50.5 is greater than the cumulative frequency 36, but less than the next cum. Freq. 55 corresponding to x = 4. All the 19 items (from 37, to 55) have the same variate 4. And 50.5 item is also one of those 19 item.

∴ Median = 4.

(C) For Continuous Series (Grouped Frequency Distribution)

We are to determine the particular class in which the value of the median lies. by using the formula $\frac{n}{2}$ (and not by $\frac{N+1}{2}$, as in continuous series $\frac{N}{2}$ divides the area of the curve into two equal parts). After locating median, its magnitude is measured by applying the formula interpolation given below:

$$\text{Median} = l_1 + \frac{l_2 - l_1}{f_m}(m - c), \; \textit{where } m = \frac{N}{2}$$

$$\left[\text{or median} = l_1 + \frac{m-c}{f_m} \times i, \text{where } i = l_2 - l_1\right]$$

Where l_1 = lower limit of the class in which median lies,

l_2 = Lower limit of the class in which median lies.

f_m = the frequency of the class in which median falls.

m = middle item (i.e., item at which median is located or $\frac{N}{2}$ th item).

C = cumulative frequency less than type of the class preceding the median class,

[*Note* : The above formula is based on the assumption that the frequencies of the class-interval in which median lies are uniformly distributed over the entire class-intrerval]

Remember :

In calculating median for a group frequency distribution, the class-intervals must be in continuous forms. If the class-intervals are given in discrete forms. They are to be converted first into continuous or class-boundaries form and hence to calculate median, apply usual formula.

Example 28 : Find the median and median-class of the data given below :—

Class-boundaries	Frequency
15–25	4
25–35	11
35–45	19
45–55	14
55–65	0
65–75	2

Solution :

Table : Calculation of Median

Class-boundaries	Frequency	Cumulative frequency
15–25	4	4
25–35	11	15
35–45	19	34
45–55	14	48
55–65	0	48
65–75	2	50 (= N)

Median = value of $\frac{N}{2}^{th}$ item = value of $\frac{50}{2}^{th}$ item = value of 25th item, which is greater than cum. Freq. 15.

So median lies in the class 35–45.

Now. Median = $l_1 + \frac{l_2 - l_1}{f}(m - c)$, where l_1 = 35, l_2 = 45, f = 19, m = 25, c = 15

$= 35 + \frac{45-35}{19}(25-15) = 35 + \frac{10}{19} \times 10 = 35 + 5.26 = 40.26$

required median is 40.26 and median-class is (35 – 45).

Example 29: Calculate the median of the table given below :

Class interval :	0–10	10–20	20–30	30–40	40–50
Frequency :	5	4	6	3	2

Solution :

Table : Calculation of Median

C.I	f	c.f
0–10	5	5
10–20	4	9
20–30	6	15
30–40	3	18
40–50	2	20 (= N)

median = value of $\frac{N}{2}^{th}$ term = value of $\frac{20}{2}$ $(=10)^{th}$ term, median class is (20–30).

Median = $l_1 + \frac{l_2 - l_1}{f}\left(\frac{N}{2} - c\right)$

$= 20 + \frac{30-20}{6}(10–9)$

$= 20 + \frac{10}{6} = 20 + 1.67 = 21.67$

5.1.5.2. Calculation of Median from Discrete Grouped Distribution

If the class intervals of grouped frequency distribution are in discrete form, at first they are to be converted into class-boundaries and hence to find median by applying usual formula. The idea will be clear from the following example.

Example 30 : Marks obtained by 62 students in English are as follows:—

Marks	No. of students
10–19	5
20–29	8
30–39	14
40–49	20
50–59	11
60–69	4
Total	62

Compute median class and median.

Solution :

The class intervals are in discrete form. They are to be converted to class boundaries first, which is shown below :

Table : Calculation of Median

Class boundaries	frequecy	Cumulative Frequency
9.5 – 19.5	5	5
19.5–29.5	8	13
29.5–39.5	14	27
39.5–49.5	20	47
49.5–59.5	11	58
59.5–69.5	4	62 (N)

Median = value of $\frac{N^{th}}{2}$ term = values of $\frac{62^{th}}{2}$ term or value of 31st term

∴ Median lies in (39.5 – 49.5)

Now median = $l_1 + \frac{l_2 - l_1}{f_m}(m - c)$, here $l_1 = 39.5$, $l_2 = 49.5$, $f_m = 20$, m = 31, c = 27

Median = $39.5 + \frac{10}{20}(31 - 27) = 39.5 + \frac{1}{2} \times 4 = 39.5 + 2 = 41.5$ marks.

Calculation of median from cumulative frequency distribution

In this case at first cumulative frequency is to be converted into general group frequency distribution. Then applying usual formula median is to be calculated.

Example 31 : Compute median from the table given below :

Marks	No. of students(f)
less than 10	3
less than 20	8
less than 30	17
less than 40	20
less than 50	22

Solution : The general group frequency distribution is as follows : —

Table : Calculation of Median

Marks	Students(f)	c.f
0–10	3	3
10–20	5(=8-3)	8
20–30	9 (=17-8)	17
30–40	3 (= 20-17)	20
40–50	2 (=22–20)	22 (N)

median = value of $\frac{N^{th}}{2}$ term = value of $\frac{22^{th}}{2}$ term = value of 11th term

$\therefore$ median class is (20 – 30)

$\therefore$ median = $l_1 + \frac{l_2 - l_1}{f_m}(m - c)$,

$= 20 + \frac{10}{9}$ (11 – 8), *here* $l_1 = 20$ $l_2 = 30$, $f_m = 9$, m = 11, c = 8

$= 20 + \frac{10}{9} = 20 + 3.33 = 23.33$ marks.

Note : If the cumulative frequency distribution is given in 'more than type' form then also the same procedure is to be followed.

Example 32 : Calculate the median of the frequency distributions

Marks :	1–20	21–40	41–60	61–80	81–100
No. of students :	3	5	9	3	2

Solution : The class intervals: are in discrete forms, so they are to be made in class boundaries at first

Table : Calculation of Median

Class boundaries	f	c.f
0.5–20.5	3	3
20.5–40.5	5	8
40.5–60.5	9	17
60.5—80.5	3	20
80.5–100.5	2	22 (=N)

Median = value of $\frac{22}{2}$ th term = values of 11th term $\therefore$ Median class is (40.5 – 60.5)

$$\text{Median} = l_1 + \frac{l_2 - l_1}{f_m}\left(\frac{N}{2} - c\right) = 40.5 + \frac{60.5 - 40.5}{9}(11 - 8)$$

$= 40.5 + \frac{20}{9} \times 3 = 40.5 + 6.67 = 47.17$ marks.

Calculation of median from open ends class intervals :

Since the first and last class intervals are not required in computing median, so in case of open end class-intervals median is calculated by usual process.

For example, in the above example it the lower-limit of first class interval (i.e.0) and upper limit of last class (i.e. 5) are not given question, there would be no difficulty to compute median.

In case of open end class-intervals, median is preferred than A.M. as average

Finding of missing frequency

The idea of finding missing frequency will be clear from the following example.

Example 33 : An incomplete frequency distributions given below :

Marks	No. of students(f)
10–20	3
20–30	5
30–40	——
40–50	3
50–60	1

It is given that median of the above distribution is 32.5 marks. Find the missing frequency.

Solution :

Table : Calculation of Median

Marks	f	c.f
10–20	3	3
20–30	5	8
30–40	f_3	$8 + f_3$
40—50	3	$11 + f_3$
50–60	1	$12 + f_3$

Here Median = 32.5 (given), so median class is (30–40).

Let f_3 be the missing frequency, $\frac{N}{2} = \frac{12+f_3}{2}$ = 6 + f_3/2 = m , c = 8, l_1 = 30 1_2=40, $f_m = f_3$

From the formula, med. = $l_1 + \frac{l_2 - l_1}{f_m}(m - c)$

We get, $32.5 = 30 + \frac{40-30}{f_3}\left(6 + \frac{f_3}{2} - 8\right)$ or, $2.5 = \frac{10}{f_3}\left(\frac{f_3}{2} - 2\right)$ or, 2.5 f_3 = 5 f_3 – 20 or $f_3 = \frac{20}{2.5} = 8$.

Advantages of Median :

(i) The median, unlike the mean, is unaffected by the extreme values of the variable.

(ii) It is easy to calculate and simple to understand, particularly in a series of individual observations a discrete series.

(iii) It is capable of further algebraic treatment. It is used in calculating mean deviation.

(iv) It can be located by inspection, after arranging the data in order of magnitude.

(v) Median can be calculated even if the items at the extreme are not known, but if we know the central items and the total number of items.

(vi) It can be determined graphically.

Disadvantage of Median :

(i) For calculation, it is necessary to arrange the data; other averages do not need any such arrangement.

(ii) It is amenable to algebraic treatment in a limited sense, Median cannot be used to calculate the combined median of two or more groups, like mean.

(iii) It cannot be computed precisely when it lies between two items.

(iv) Process involved to calculate median in case of continuous series is difficult to follow.

(v) Median is affected more by sampling fluctuations than the mean.

SELF EXAMINATION QESTIONS :

1. Define median, Mention merits and demerits of median.

2. Find median of the following numbers :

(i) 38, 56, 31, 70, 41, 62, 53, 57 [Ans. 54.5]

(ii) 14, 15, 30, 40, 10, 25, 20, 35 [Ans. 22.5]

(iii) 25, 1275, 748, 162, 967, 162 [Ans. 455]

3. Of the numbers 78, 82, 36, 38, 50, 72, 68, 70, 64 find median. [Ans. 68]

4. The heights (in cm) of few students are as follows :

69, 75, 72, 71, 73 , 74, 76, 75, 70

Find second quartile. [Ans. Q_2 = 72.5 cm]

5. Find the median of the following numbers :

6, 4, 3, 6, 5, 3, 3, 2, 4, 3, 4, 3, 3, 4, 3, 4, 2, 2, 4, 3, 5, 4, 3, 4, 3, 3, 4, 1, 1, 2, 3. [Ans. 3]

6. Find the median of the following distribution :

x	y
1	22
2	31
3	40
4	42
5	24
6	12

[Ans.3]

7. Find the median class and median from the table given :

(i)

C.I	*Frequency*
0–10	5
10–20	4
20–30	6
30–40	3
40–50	2

[Ans. (20–30); 21.67]

(ii)

Score	*Frequency*
5–10	4
10–15	7
15–20	10
20–25	12
25–30	8
30–35	3
Total	44

[Ans.(20 –25); 24.17 score]

8. Find the following distribution find median class and median :

Score	Frequency
30–39	1
40–49	4
50–59	14
60–69	20
70–79	22
80–89	12
90–99	2

[Ans. (59.5–69.5); 68.75 score]

9. Calculate median of table given :

	Marks	Students	Marks	Students
(i)	Less than10	5	(ii) Less than 45	20
	Less than 20	9	Less than 40	17
	Less than 30	15	Less than 35	12
	Less than 40	18	Less than 30	5
	Less than 50	20	Less than 25	2

[Ans. (i) 21.67 marks, (ii) 33.57 marks]

10. In the following frequency distribution one frequency is missing. It is given that median of the distribution is 53.5, find the missing frequency.

Variate	Frequency
20–30	8
30–40	5
40–50	f_3
50–60	20
60–70	10
70–80	4

[Ans. 12]

[Note. That the class interval are unequal.]

11. The expenditure of 1000 families are as follows :

Expenditure (₹)	No. of families
40–59	50
60–79	––
80–99	500
100–119	––
120–139	50

In the above table median is ₹ 87.50. Find the missing frequency.

5.1.6. MODE

Definition :

Mode is the value of the variate which occurs most frequently. It represents the most frequent value of a series. In other words **Mode** is the value of the variable which has the highest frequency.

When one speak of the 'average student', we generally mean the modal wage, the modal student. If we say that the modal wages obtained by workers in a factory are ₹ 70, we mean that the largest number of workers get the same amount. As high as ₹ 100 and as low as ₹ 50 as wages are much less frequented and they are non-modal.

Calculation.

Mode cannot be determined in a series of individual observations unless it is converted to a discrete series (or continuous series). In a discrete series the value of the variate having the maximum frequency is the modal class. However, the exact location of mode is done by interpolation formula like median.

Location of modal value in case of discrete series is possible if there is concentration of items at on point. If again there are two or more values having same maximum frequencies, (i.e. more concentration), it becomes difficult to determine mode. Such items are known as bimodal, tri-modal or multi-modal accordingly as the items concentrate at 2, 3 or more values.

(A) For. Individual Observations

The individual observations are to be first converted to discrete series (if possible).

Then the variate having the maximum will be the mode.

Example 34 : Calculate mode from the data (given) :

(Marks) : 10, 14, 24, 27, 24, 12, 11, 17.

Marks	Frequency
10	1
11	1
12	1
14	1
17	1
24	2
27	1

(Individual observation are converted into a discrete series)

Here marks 24 occurs maximum number of times, i.e. 2. Hence the modal marks is 24, or mode = 24 marks.

Alternatively :

Arranging the numbers : 10, 11, 12, 14, 17, (24, 24) 27.

Now 24 occurs maximum number of times, i.e. 2. ∴ Mode = 24 marks.

[*Note*. When there are two or more values having the same maximum frequency, then mode is ill-defined. Such a sense is known as bimodal or multi-modal as the case may be.]

Example 35 : Compute mode from the following data.

Marks obtained : 24, 14, 20, 17, 20, 14.

Marks	Frequency
14	2
17	1
20	2
24	1

[Here 14 occurs 2 times (max.) and 20 occurs 2 times (max.)

∴ mode is ill-defined.]

(B) For Simple Frequency Distribution Discrete Series.

To Find the mode from the following Table :

Height (in inches)	No. of Persons
57	3
59	5
61	7
62	10
63	20
64	22
65	24
66	5
67	2
69	2

Frequencies given below, in column (1) are grouped by two's in column (2) and (3) and then by three's in columns (4), (5), and (6). The maximum frequency in each column is marked by **Bold Type**. We do not find any fixed point having maximum frequency but changes with the change of grouping. In the following table, the sizes of maximum frequency in respect of different columns are arranged.

Grouping Table

Inches	Grouping Table Height Frequency						
	No. of persons	(1)	(2)	(3)	(4)	(5)	(6)
57	3						
		8					
59	5			15			
			12				
61	7				22		
		17					
62	10						37
			30	52			
63	20						
		42			66		
64	22						
			46				
65	24						51
		29		31			
66	5		7		9		
67	2						
		4					
69	2						

Analysis Table

Column	Sizes of items having maximum frequency				
1				65	
2		63	64		
3			64	65	
4	62	63	64		
5		63	64	65	
6			64	65	66
No. of items	1	3	5	4	1

From the above table, we find 64 is the size of the item which is most frequented. The mode is, therefore, located at 64.

[*Note*. At glance from column (1) one might think that 65 is the mode since it contains maximum frequency. This impression is corrected by the process of grouping . So it is not advisable to locate the mode merely by inspection.]

(C) For continuous Series.

By inspections or by preparing Grouping Table and Analysis Table, ascertain the modal class. Then to find the exact value of mode, apply the following formula.

$$\text{Mode} = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times i.$$

Where, l = lower class-boundary of modal class

f_1 = frequency of modal class.

f_0 = frequency of the class preceding modal class.

f_2 = frequency of the class succeeding the modal class.

i = size of the class- interval of modal class.

Note : the above formula may also be expressed as follows :

Mode = $l + \frac{f_1 - f_0}{(f_1 - f_0) + (f_1 - f_2)} \times i = l + \frac{\Delta_1}{\Delta_1 + \Delta_2} \times i.$ Where $\Delta_1 = f_1 - f_0$ ∴ $\Delta_2 = f_1 - f_2$.

Example 36 : Compute mode of the following distribution.

Marks	No. of students
10–20	5
20–30	8
30–40	12
40–50	16
50–60	10
60–70	8

Table : Calculation of Mode

Marks	No. of students
10–20	5
20–30	8
30–40	$12 \rightarrow f_0$
40–50	$16 \rightarrow f_1$
50–60	$10 \rightarrow f_2$
60–70	8

From the table it is clear that the maximum frequency is 16th : modal class is (40–50)

Here l = 40, f_0 = 12, f_1 = 16, f_2 = 10 (marked in table), i = 10 (= 50 –40)

$$\text{Mode} = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times i$$

$$= 40 + \frac{16 - 12}{2 \times 16 - 12 - 10} \times 10$$

$$= 40 + \frac{4}{32 - 22} \times 10$$

$$= 40 + \frac{4}{10} \times 10 = 40 + 4 = 44 \text{ marks.}$$

Alternatively, $D_1 = f_1 – f_0 = 16 – 12 = 4$,

$D_2 = f_1 – f_2 = 16 – 10 = 6$, i = 10, l = 40

$$\text{Mode} = 40 + \frac{4}{4+6} \times 10 = 40 + \frac{4}{10} \times 10$$

= 40 + 4 = 44 marks.

Calculation of Mode From discrete group frequency distribution.

In such cases at first class boundaries are to be formed for applying formula.

Example 37: Compute mode from the following frequency distribution :

Marks	No. of students
50–59	5
60–69	20
70–79	40
80–89	50
90–99	30
100–109	6

The class intervals which are in discrete form are first converted into class boundaries.

Table : Calculation of mode

Class boundaries	Frequency
49.5–59.5	5
59.5–69.5	20
69.5–79.5	40
79.5–89.5	50
89.5–99.5	30
99.5–109.5	6

Now modal class is (79.5 – 89.5), since this class has the highest frequency.

Here l = 79.5, f_0 = 40, f_1 = 50, f_2 = 30, i = 10

$$\text{Mode} = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times i = 79.5 + \frac{50-40}{100-40-30} \times 10$$

$$= 79.5 + \frac{10}{30} \times 10 = 79.5 + \frac{10}{3} = 79.5 + 3.33 = ₹\ 82.83.$$

Calculation of mode from cumulative frequency distribution :

Example 38 : From the following cumulative frequency distribution of marks of 22 students in Accountancy, calculate mode :

Marks	below 20	below 40	below 60	below 80	below 100
No. of students	3	8	17	20	22

Solution :

At first we are to transfer the above cumulative frequency distribution into a equal group frequency distribution and hence to calculate mode.

Table : Calculation of mode

Marks	students(f)
0–20	3
20–40	5 (= 8 –3)
40–60	9(=17—8)
60–80	3 (=20–17)
80–100	2 (=22–20)

Modal class is (40–60), as this class has highest frequency.

Here l = 40, f_0 = 5, f_1 = 9, f_2 = 3, i = 20

$$\text{Mode } l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times i = 40 + \frac{9-5}{2 \times 9 - 5 - 3} \times 20 = 40 + \frac{4}{10} \times 20 = 40 + 8 = 48 \text{ marks.}$$

Calculation of missing frequency :

Example 39 : Mode of the given distribution is 44, find the missing frequency

Marks	10–20	20–30	30–40	40–50	50–60	60–70
No. of students	5	8	12	––	10	8

Solution :

Since mode is 44, so modal class is 40–50.

Table : Showing the Frequency Distribution

Marks	Frequency(f)
10–20	5
20–30	8
30–40	12
40–50	––
50–60	10
60–70	8

let the missing frequency be f_1

Now mode = $l+\frac{f_1-12}{2f_1-12-10}\times 10$

or, 44 = $40+\frac{f_1-12}{2f_1-22}\times 10$

or, $4=\frac{f_1-12}{2f_1-22}\times 10$ or, f_1 = 16 (on reduction)

Miscellaneous examples :

1. If two variates x and y are related by 2x = 3y – 1, and mean of y be 9 ; find the mean of x.

2x = 3y –1 or, $2\overline{x}=3\overline{y}-1$ or, $2\overline{x}=3\times 9-1=26$ or, $\overline{x}=13$

2. If 2u = 5x is the relation between two variables x and u and harmonic mean of x is 0.4, find the harmonic mean of u.

$u=\frac{5x}{2}$ or, $u=\frac{5}{2}\times 0.4=1.0$ $\therefore$ reqd. H.M is 1.0

3. The relation between two variables x and y is 3y – 2x + 5 = 0 and median of y is 40, find the median of x.

From $3y-2x+5=0$ we get, $x=\frac{3}{2}y+\frac{5}{2}$. As the median is located by position, so median of x is $\frac{3}{2}.\ 40+\frac{5}{2}=62.5$

4. Mode of the following frequency distribution is 24 and total frequency is 100. Find the values of f_1 and f_2.

C.I :	0 –10	10–20	20–30	30–40	40–50
Frequency :	14	f_1	27	f_2	15

Mode is 24, so modal class is (20–30). From the formula of mode we find. $24 = 20 + \frac{27 - f1}{54 - f_1 - f_2} \times 10$ or,

$$4 = \frac{270 - 10f_1}{54 - (f_1 + f_2)}$$

or, $4 = \frac{270 - 10f_1}{54 - 44} = \frac{270 - 10f_1}{10}$

$14 + f_1 + 27 + f_2 + 15 = 100$

or, $f_1 + f_2 = 100 - 56 = 44$ ....(1)

or, $40 = 270 - 10f_1$

or, $10f_1 = 230$ or, $f_1 = 23$. From (1), $f_2 = 44 - 23 = 11$ $\therefore$ $f_1 = 23,\ f_2 = 11$

5. The following are the monthly salaries in rupees of 20 employees of a firm :

130	125	110	100	80	76	98	103	122	66
145	151	65	71	118	140	116	85	95	151

The firm gives bonuses of ₹ 10, 15, 20, 25 and 30 for individuals in the respective salary group : exceeding ₹ 60 but not excedding ₹ 80, exceeding ₹ 80 but not exceeding ₹ 100 and so on up to exceeding ₹ 140 but not exceeding ₹ 160. Find the average bonus paid per employee.

Solution:

From the monthly salaries of the employees, we find the number of employees lying in the salary groups mentioned as follows :

Table : Calculation of average bonus

Salary (₹)					bonus	
				f	x	fx
Exceeding 60 but not exceeding			80	5	10	50
"	80	"	100	4	15	60
"	100	"	120	4	20	80
"	120	"	140	4	25	100
"	140	"	160	3	30	90
Total				20		380

$$\text{A.M. } (\bar{x}) = \frac{\sum fx}{\sum f} = \frac{380}{20} = \text{Rs. } 19.$$

6. Marks obtained by 30 students in History of a Test Examination 2012 of some school are as follows :

34	36	10	21	31	32	22	43	48	36
48	22	39	26	34	39	10	17	47	38
40	51	35	52	41	32	30	35	53	23

construct a frequency table with class intervals 10–19. 20–29 etc. Calculate the median and mode from the frequency distribution.

Solution:

Table : Construction of frequency table and hence calculation of median and mode.

Marks	f	cf	class boundaries
10–19	3	3	9.5 – 19.5
20–29	5	8	19.5 – 29.5
30 – 39	13	21	29.5 – 39.5
40 – 49	6	27	39.5 –49.5
50 – 59	3	30	49.5 – 59.5

Median = value of $\frac{N}{2}$th i.e., $\frac{30}{2}$ i.e. 15th term

So median class is (29.5 – 39.5)

$$\therefore \text{ median } = 29.5+\frac{39.5-29.5}{13}(15-8) \; = 29.5+\frac{10}{13}\times 7$$

= 29.5 + 5.38 = 34.88 marks

Highest frequency is 13 (= f_1), f_0 = 5, f_2 = 6

$$\therefore \text{Mode } = l+\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\times i = 29.5+\frac{13-5}{2\times 13-5-6}\times 10$$

$$= 29.5+\frac{8}{15}\times 10 = 29.5+5.33 \; = 34.83 \text{ marks.}$$

Advantages of mode :

(i) It can often be located by inspection.

(ii) It is not affected by extreme values. It is often a really typical value.

(iii) It is simple and precise. It is an actual item of the series except in a continuous series.

(iv) Mode can be determined graphically unlike Mean.

Disadvantages of mode :

(i) It is unsuitable for algebraic treatment.

(ii) When the number of observations is small, the Mode may not exist, while the Mean and Median can be calculated.

(iii) The value of Mode is not based on each and every item of series.

(iv) It does not lead to the aggregate, if the Mode and the total number of items are given.

5.1.7. Empirical Relationship among Mean, Median and Mode

A distribution in which the values of Mean, Median and Mode coincide, is known symmetrical and if the above values are not equal, then the distribution is said asymmetrical or skewed. In a moderately skewed distribution, there is a relation amongst Mean, Median and Mode which is as follows :

Mean – Mode = 3 (Mean – Median)

If any two values are known, we can find the other.

Example 40 : In a moderately asymmetrical distribution the mode and mean are 32.1 and 35.4 respectively. Calculate the Median.

From the relation, we find

3 Median = 2 Mean + Mode

or 3 Median = 2 × 35.4 + 32.1 = 70.8 + 32.1 = 102.9

$\therefore$ Median = 34.3

SELF EXAMINATION QUESTIONS :

1. Define mode. Mention the advantages and disadvantages of mode.

2. Calculate the mode of the following numbers :

(i) 25, 1275, 748, 169, 876, 169 [Ans. 169]

(ii) 4, 3, 2, 5, 3, 4, 5, 1, 7, 3, 2, 1 [Ans. 3]

(iii) 69, 75, 57, 70, 71, 75, 76 [Ans. 75]

(iv) 1, 3, 4, 7, 9, 10, 11, 13, 14, 16 [Ans. 11]

3. Find the mode of the numbers :

7, 4, 3, 5, 6, 3, 3, 2, 4, 3, 3, 4, 4, 2, 3 [Ans. 3]

4. Find the mode of the following frequency distribution :-

x	f
0	5
1	22
2	31
3	43
4	51
5	40
6	35
7	15
8	3

[Ans. 4]

5. Compute mode from the following frequency distribution :

Marks	Students
0–10	3
10–20	7
20–30	10
30–40	6
40–50	2

[Ans. 25 marks]

(ii)

Score	Frequency
25–30	3
30–35	5
35–40	6
40–45	10
45–50	9

[Ans. ₹ 44]

6. Calculate mode of the distribution given below :

(i)

Marks	No. of students
less than 10	5
less than 20	9
less than 30	15
less than 40	18
less than 50	20

(ii)

Wages	No. of workers
0 and above	50
20 and above	45
40 and above	34
60 and above	16
80 and above	6
100 and above	0

[Ans. (i) 24 marks, (ii) ₹ 49.33]

7. Daily wages of 100 worker are given in the table :

Daily Wages (₹)	No. of workers
2–3	5
4–5	8
6–7	12
8–9	10
10–11	7

Compute the modal value.

OBJECTIVE QUESTIONS

1. What is the sum of deviations of a variates from their A.M.? [Ans. zero]

2. Write the relation of AM. G.M. and H.M. [Ans. AM ^{3} G.M ^{3} H.M]

3. For a pair of variates, write the relation of AM. G.M and H.M [Ans. $(GM)^2$ = AM × H.M]

4. Write the emperical relation of mean, median and mode. [Ans. Mean-mode = 3 (Mean – median)]

5. In case of open end class intervals frequency distribution to calculate average, which is most appropriate average? [Ans. Median]

6. Find A.M and mode of : 7, 4, 10, 15, 7, 3, 5, 2, 9, 12 [Ans. 7.4 ; 7]

7. Find G.M. of 3, 12, 48 [Ans. 12]

8. For a symmetry distribution mode and A.M. are respectively ₹ 12.30 and ₹ 18-48 ; find median of the distribution. [Ans. 16.42]

9. The mean marks of 100 students was found to be 40. Later on it was discovered that marks 53 was misread as 83. Find the corrected mean marks [Ans. 39.70]

10. A.M. of 7, x – 2, 10, x + 3 is 9 find x [Ans. 9]

11. Find G.M. of 8 observations : 2 occuring 4 times, 4 occuring twice 8 and 32 occuring once each. [Ans. 4]

12. Find H.M. of the observations $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ and $\frac{1}{8}$ [Ans. $\frac{1}{5}$]

13. If the means of two groups of m and n observations are 40 and 50 respectively and the combined group mean is 42, find the ratio m : n. [Ans. 4 :1]

14. Find mean and mode of the 9 observations 9, 2, 5, 3, 5, 7, 5, 1, 8 [Ans. 5, 5]

15. If two groups have number of observations 10 and 5 and means 50 and 20 respectively, find the grouped mean. [Ans. 40]

16. Two variables x and y are related by $y = \frac{x-5}{10}$ and each of them has 5 observation. If mean of x is 45, find the mean of y. [Ans. 4]

17. Find H.M. of $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ..., \frac{n}{n+1}$, occuring with frequencies 1, 2, 3, …., n respectively [Ans. 1]

[**Hints :** H.M. $= \frac{1+2+3+....+n}{1.2+2.\frac{3}{2}+3.\frac{4}{3}+.....+n.\frac{n+1}{n}} = \frac{n(n+1)/2}{2+3+4+....+n+1} = \frac{n(n+1)/2}{n(n+1)/2} = 1$

18. If the relation between two variables x and y be 2x + 5y = 24 and mode of y is 4, find mode of x. [Ans. 2]

19. Find median of the 10 observations 9, 4, 6, 2, 3, 4, 4, 6, 8, 7 [Ans. 5]

20. The mean of 10 observations was found to be 20. Later on one observation 24 was wrongly noted as 34. Find the corrected mean. [Ans. 19]

21. Prove that for two numbers 2 and 4, AM × HM = $(G.M.)^2$.

22. If the relation between two variables x and y is 2x + 3y = 7, and median of y is 2, find the median of x. [Ans. $\frac{1}{2}$]

23. If two groups of 50 and 100 observations have means 4 and 2 respectively, find the mean of the combined group. [Ans. $3\frac{2}{3}$]

24. If a variable x takes 10 values 1, 2, 3, …., 10 with frequency as its values in each case, then find the arithmetic mean of x. [Ans. 7]

[**Hints :** A.M. $= \frac{1^2+2^2+3^2+......+10^2}{1+2+3+.....+10}$ & etc.]

25. If first of two groups has 100 items and mean 45 and combined group has 250 items and mean 5/, find the mean of second group. [Ans. 55]

26. Find the median of the following distribution

Weight (kg)	:	65	66	67	68	[Ans. 67]
No. of students	:	5	15	17	4	

27. Find G.M. of 3, 6, 24, 48 [Ans. 12]

28. A.M. of two numbers is 25 and their H.M. is 9, find their G.M. [Ans. 15]

29. The means of samples of sizes 50 and 75 are 60 and x respectively. If the mean of the combined group is 54, find x. [Ans. 50]

30. Find the median of the given distribution :

Value (x)	:	1	2	3	4	[Ans. 3]
Frequency (f)	:	7	12	18	4	

31. If each of 3, 48 and 96 occurs once and 6 occurs twice verify that G.M. is greater than H.M.

32. Find G M. of 1, 2, 3, $\frac{1}{2}, \frac{1}{3}$. What will be G.M. if '0' is added to above set of velues?

[Ans. 1 ; 0]

33. The G.M. of a, 4, 6 is 6, find a [Ans. 9]

34. A.M. of a variable x is 100, find the mean of the variable 2x – 50. [Ans. 150]

35. The variable x and y are given by y = 2x + 11. If the median of x is 3, find the median of y. [Ans. 17]

5.2 QUARTILE DEVIATION

Quartiles are such values which divide the total number of observations into 4 equal parts. Obviously, there are 3 quartiles—

(i) First quartile (or Lower quartile): Q_1

(ii) Second quartile, (or Middle quartile) : Q_2

(iii) Third quartile (or Upper quartile): Q_3

The number of observations smaller than Q_1, is the same as the number lying between Q_1 and Q_2, or between Q_2 and Q_3, or larger than Q_3. For data of continuous type, one-quarter of the observations is smaller than Q_1, two-quarters are smaller than Q_2, and three-quarters are smaller than Q_3. This means that Q_1, Q_2, Q_3 are values of the variable corresponding to 'less-than' cumulative frequencies N/4, 2N/4, 3N/4 respectively. Since, 2N/4 = N/2, it is evident that the second quartile Q_2 is the same as median.

$$Q_1 < Q_2 < Q_3; \qquad Q_2 = \text{Median.}$$

Quartiles are used for measuring central tendency, dispersion and skewness. For instance, the second quartile Q_2 is itself taken as a measure of central tendency, where it is known as Median.

Quartile deviation is defined as half the difference between the upper and the lower quartiles.

$$\text{Quartile Deviation} = \frac{Q_3 - Q_1}{2}$$

The difference Q3- Q1 being the distance between the quartiles can also be called **inter quartile range;** half of this Semi- inter quartile Range. Thus the name 'Semi - inter quartile Range' itself gives the definition of Quartile Deviation.

Quartile Deviation (Q.D.) is dependent on the two quartiles and does not take into account the variability of the largest 25% or the smallest 25% of observations. It is therefore unaffected by extreme values. Since in most cases the central 50% of observation tend to be fairly typical, Q.D. affords a convenient measure of dispersion. It can be calculated from frequency distributions with open-end classes. Q.D. is thus superior to Range in many ways. Its unpopularity lies in the fact that Q.D. does not depend on the magnitudes of all observations. The calculation of Q.D. only depends on that of the two quartiles, Q1 and Q3 which can be found from a cumulative frequency distribution using simple interpolation.

Example 41 : Calculate the quartile deviation from the following:

Class interval	10-15	15-20	20-25	25-30	30-40	40-50	50-60	60-70	Total
Frequency	4	12	16	22	10	8	6	4	82

Solution :

In order to compute Quartile Deviation, we have to find Q_1 and Q_3 i.e. values of the variable corresponding to Cumulative frequencies N/4 and 3N/4. Here, total frequency N = 82. Therefore, N/4 = 20.5 and 3N/4 = 61.5

Cumulative Frequency Distribution

Class Boundary	**Cumulative Frequency (less - than)**
10	0
15	4
20	16
$Q_1 \rightarrow$	$\leftarrow$N/4 = 20.5
25	32
30	54
$Q_3 \rightarrow$	$\leftarrow$3N/4 = 61 .5
40	64
50	72
60	78
70	82 = N

Applying simple interpolation,

$$\frac{Q_1 - 20}{25 - 20} = \frac{20.5 - 16}{32 - 16}$$

or, $$\frac{Q_1 - 20}{5} = \frac{4.5}{16}$$

or, $$Q_1 - 20 = \frac{4.5}{16} \times 5 = 1.4$$

or, $$Q_1 = 20 + 1.4 = 21.4$$

Similarly

$$\frac{Q_3 - 30}{40 - 30} = \frac{61.5 - 30}{64 - 54}$$

or, $$\frac{Q_3 - 30}{10} = \frac{7.5}{10}$$

or, $Q_3 - 30 = \frac{7.5}{10} \times 10 = 7.5$

or, $Q_3 = 30 + 7.5 = 37.5$

Quartile Deviation = $\frac{Q_3 - Q_1}{2} = \frac{37.5 - 21.4}{2} = 8.05$

Example 42 : Calculate the appropriate measure of dispersion from the following data :

Wages in rupees per week	No. of wages earners
less than 35	14
35-37	62
38-40	99
41-43	18
over 43	7

Solution: Since there are open-end classes in the frequency distribution. Quartile deviation would be the most appropriate measure of dispersion for the data. So, we have to determine the quartiles Q_1 and Q_3 which can be done from cumulative frequency distribution using simple interpolation.

Table : Cumulative Frequency Distribution

Wages (₹) per week	Cumulative frequency
34.5	14
$Q_1 \rightarrow$	$\leftarrow N/4 = 50$
37.5	76
$Q_3 \rightarrow$	$\leftarrow 3N/4 = 150$
40.5	175
43.5	193
......	200 = N

Applying Simple interpolation,

$$\frac{Q_1 - 34.5}{37.5 - 34.5} = \frac{50 - 14}{76 - 14} \quad \text{and} \quad \frac{Q_3 - 37.5}{40.5 - 37.5} = \frac{150 - 76}{175 - 76}$$

Solving Q_1 = ₹ 36.24 $\quad$ Q_3 = ₹ 39.74

Therefore, Quartile Deviation = $\frac{(39.74 - 36.24)}{2} = 1.75$

5.3 MEASURES OF DISPERSION

5.3.1. DISPERSION

A measure of dispersion is designed to state the extent to which individual observations (or items) vary from their average. Here we shall account only to the amount of variation (or its degree) and not the direction.

Usually, when the deviation of the observations form their average (mean, median or mode) are found out then the average of these deviations is taken to represent a dispersion of a series. This is why measure of dispersion are known as *Average of second order*. We have seen earlier that mean, median and mode, etc. are all *averages of the first order*.

Measures of dispersion are mainly of two types–

(A) Absolute measures are as follows :

(i) Range, (ii) Mean deviation (or Average deviation), (iii) Standard deviation

(B) Among the Relative measures we find the following types :

(i) Coefficient of dispersion. (ii) Coefficient of variation.

5.3.2. Absolute and Relative measures :

If we calculate dispersion of a series, say, marks obtained by students in absolute figures, then dispersion will be also in the same unit (i.e. marks). This is *absolute dispersion*. If again dispersion is calculated as a ratio (or percentage) of the average, then it is *relative dispersion*.

5.3.2.1. RANGE :

For a set observations, range is the difference between the extremes, i.e.

Range = Maximum value – Minimum value

Illustration 43. The marks obtained by 6 students were 24, 12, 16, 11, 40, 42. Find the Range. If the highest mark is omitted, find the percentage change in the range.

Here maximum mark = 42, minimum mark = 11.

$\therefore$ Range = 42 – 11 = 31 marks

If again the highest mark 42 is omitted, then amongst the remaining. Maximum mark is 40. So, range (revised) = 40 – 11 = 29 marks.

Change in range = 31 – 29 = 2 marks.

$\therefore$ Reqd. percentage change = 2 ÷ 31 × 100 = 6.45%

Note : Range and other obsolute measures of dispersion are to be expressed in the same unit in which observations are expressed.

For grouped frequency distribution :

In this case range is calculated by subtracting the lower limit of the lowest class interval from the upper limit of the highest.

Example 44 : For the following data calculate range :

Marks	Frequency
10–15	2
15–20	3
20–25	4
25–30	1

Here upper limit of the highest class interval = 30

And lower limit of the first class interval = 10

$\therefore$ Range = 30 – 10 = 20 marks

Note : Alternative method is to subtract midpoint of the lowest class from that of the highest. In the above case, range = 27.5 – 12.5 = 15 marks.

In practice both the methods are used.

Coefficient of Range :

The formulae of this relative measure is

$$\frac{\text{difference of extreme value i.e. range}}{\text{sum of extreme vales}}$$

In the above example, Coefficient of range $= \frac{30-10}{30+10} = \frac{20}{40} = \frac{1}{2} = 0.5$

Advantages of Range : Range is easy to understand and is simple to compute.

Disadvantages of Range :

It is very much affected by the extreme values. It does not depend on all the observations, but only on the extreme values. Range cannot be computed in case of open-end distribution.

Uses of Range :

It is popularly used in the field of quality control. In stock-market fluctuations range is used.

5.3.2.2. Mean Deviation (or Average Deviation) :

Mean deviation and standard deviation, however, are computed by taking into account all the observations of the series, unlike range.

Definition :

Mean deviation of a series is the arithmetic average of the deviations of the various items from the median or mean of that series.

Median is preferred since the sum of the deviations from the median is less than from the mean. So the values of mean deviation calculated from median is usually less than that calculated from mean.

Mode is not considered, as its value is indeterminate.

Mean deviation is known as *First Moment* of dispersion.

Computation of Mean Deviation :

(a) For individual Observation (or Simple Variates)

The formula is Mean Deviation (M.D.) $= \frac{\sum |d|}{n}$

Where | d | within two vertical lines denotes deviations from mean (or median), ignoring algebraic signs (i.e., + and –).

Coefficient of Mean Deviation :

Coefficient of Mean Deviation = $\frac{\text{Mean Deviation about Mean (or Median)}}{\text{Mean (or Median)}} \times 100\%$

Steps to find M. D.

(1) Find mean or median

(2) Take deviation ignoring ± signs

(3) Get total of deviations

(4) Divide the total by the number of items.

Example 45 : To find the mean deviation of the following data about mean and median :

(₹) 2, 6, 11, 14, 16, 19, 23.

Solution :

Table : Computation of Mean Deviation.

About Mean			About Median		
Serial No.	(₹) x	Dev. From A.M. ignoring ± signs | d |	Serial No.	(₹) x	Dev. From Med. ignoring ± signs | d |
1	2	11	1	2	12
2	6	7	2	6	8
3	11	2	3	11	3
4	14	1	4	14	0
5	16	3	5	16	2
6	19	6	6	19	5
7	23	10	7	23	9
Total	––	40	Total	––	39

$$\text{A.M.} = \frac{1}{7}(2+6+11+14+16+19+23) = \frac{1}{7} \times 91 = ₹\ 13$$

Median = size of $\frac{7+1}{2}$ th iterm = size of 4th item = ₹ 14

$$\text{Mean deviation (about mean)} = \frac{|d|}{n} = \frac{40}{7} = ₹\ 5.71$$

$$\text{Mean deviation (about median)} = \frac{|d|}{n} = \frac{39}{7} = ₹\ 5.57$$

Note : The sum of deviation $\left(\sum |d|\right)$ about median is 39, less than | d | about mean (= 40). Also M.D. about median.(i.e.5.57) is less than that about mean, (i.e., 5.71)

Coefficient of Mean Deviation :

$$\text{About mean, Coefficient of M.D} = \frac{\text{M.D.}}{\text{Mean}} = \frac{5.71}{13} = 0.44 \text{ (app.)}$$

$$\text{About median, Coefficient of M. D.} = \frac{\text{M.D.}}{\text{Median}} = \frac{5.57}{14} = 0.40 \text{ (app.)}$$

(b) For Discrete Series (or Simple Frequency Distribution)

The formula for computing M.D. is

$$\text{M.D.} = \frac{\sum f_i |x_i - \bar{x}|}{\sum f_i} \quad \text{or} \quad \frac{\sum f|x-\bar{x}|}{\sum f} = \frac{\sum f|d|}{\sum f}$$

Where | d | = deviations from mean (or median) ignoring ± signs.

Steps of find M.D.

(i) Find weighed A.M. or median.

(ii) Find deviations ignoring ± signs. i.e., | d |

(iii) Get $\sum f|d|$;

(iv) Divide $\sum f|d|$ by $\sum f$

About Mean

Example 46 : To calculate mean deviation of the following series :

x (marks)	f (student)
5	6
10	7
15	8
20	11
25	8
Total	40

Find also the coefficient of dispersion.

Solution :

Table : Computation of Mean Deviation (About Mean)

Marks					Deviation From Mean (16)	from actual
X	f	(d= x – 15)	d′ = d/5	fd′	$\lvert x-\bar{x}\rvert$	$f\lvert x-\bar{x}\rvert$
(1)	(2)	(3)	(4)	(5) = 2 × (4)	(6)	(7) = (2) × (6)
5	6	–10	–2	–12	11	66
10	7	–5	–1	–7	6	42
15	8	0	0	0	1	8
20	11	5	1	11	4	44
25	8	10	2	16	9	72
Total	40	–	–	8	–	232

$$\text{A.M.} = A + \frac{\sum fd'}{\sum f} \times i = 15 + \frac{8}{40} \times 5 = 15 + 1 = 16 \text{ marks}$$

$$\text{M.D.} = \frac{\sum f|x-\bar{x}|}{\sum f} = \frac{232}{40} = 5.8 \text{ marks.}$$

$$\text{Coefficient of dispersion (about mean)} = \frac{\text{M.D.}}{\text{Mean}} = \frac{5.8}{16} = 0.363$$

About Median :

Example 47 : The same example as given above.

Table : Computation of Mean deviation (About median)

Marks X	f	Cum. Freq. c.f.	Dev. From median (15) | d |	f | d |
5	6	6	10	60
10	7	13	5	35
15	8	21	0	0
20	11	32	5	55
25	8	40 (= N)	10	80
Total	40	–	–	230

Median = value of the $\frac{40+1}{2}$ th item

= value of 20.5 th item = 15 marks.

M.D. $= \frac{\sum f|d|}{\sum f} = \frac{230}{40}$ = 5.75 marks

Coefficient of dispersion (about median) $= \frac{\text{M.D.}}{\text{Median}} = \frac{5.75}{15}$ = 0.383

(c) For Class Intervals (or Group Distribution)

Steps to compute (M.D.)

(i) Find mid-value of the class intervals

(ii) Compute weighted A.M. or median

(iii) Find | d | and f | d |

(iv) Divide $\sum f|d|$ by $\sum f$

Example 48 : Find M.D. about A.M. of the following frequency distribution :

Daily wages (₹)	No. of workers
3.50–5.50	6
5.50–7.50	14
7.50–9.50	16
9.50–11.50	10
11.50–13.50	4

Calculate also M.D. about median and hence find coefficient of mean dispersion.

Solution :

Table : Computation of M.D. about A.M.

Wages (₹)	Mid-value x	f	$d' = \frac{x-8.50}{2}$	fd′	$\lvert d \rvert = \lvert x-\bar{x} \rvert$	f | d |
3.50–5.50	4.50	6	–2	–12	3.68	22.08
5.50–7.50	6.50	14	–1	–14	1.68	23.52
7.50–9.50	8.50	16	0	0	0.32	5.12
9.50–11.50	10.50	10	1	10	2.32	23.20
11.50–13.50	12.50	4	2	8	4.32	17.28
Total	–	50	–	–8	–	91.20

$$\bar{x}\,(\text{A.M.}) = A + \frac{\sum fd'}{\sum f} \times i = 8.50 + \frac{(-8)}{50} \times 2 = 8.50 - 0.32 = 8.18$$

$$\text{M.D.} = \frac{\sum f|d|}{\sum f} = \frac{91.20}{50} = 1.824 = ₹\ 1.82$$

Table : Calculation of M.D. about median

Wages (₹)	f	c.f.	mid-value	| d |	f | d |
3.50–5.50	6	6	4.50	3.63	21.78
5.50–7.50	14	20	6.50	1.63	22.82
7.50–9.50	16	36	8.50	0.37	5.92
9.50–11.50	10	46	10.50	2.37	23.70
11.50–13.50	4	50 (N)	12.50	4.37	17.48
Total	50	–	–	–	91.70

Median = value of N/2th item = value of 50/2, i.e., 25 th item.

So median class is (7.50 – 9.50)

$$\therefore \text{ Median} = l_1 + \frac{l_2 - l_1}{f_1}(m - c) = 7.50 + \frac{9.50 - 7.50}{16}(25 - 20)$$

$$= 7.50 + \frac{2}{16} \times 5 = 7.50 + 0.625 = 8.125 = ₹\ 8.13$$

$$\text{M.D.} = \frac{\sum f|d|}{\sum f} = \frac{91.70}{50} = 1.834 = ₹\ 1.83$$

$$\text{Coeff. Of dispersion (about A.M.)} = \frac{\text{M.D.}}{\text{A.M.}} = \frac{1.824}{8.18} = 0.223 = 0.22$$

$$\text{Coeff. Of dispersion (about median)} = \frac{\text{M.D.}}{\text{Median}} = \frac{1.834}{8.13} = 0.225 = 0.23$$

Advantages of Mean Deviation :

(1) It is based on all the observations. Any change in any item would change the value of mean deviation.

(2) It is readily understood. It is the average of the deviation from a measure of central tendency.

(3) Mean Deviation is less affected by the extreme items than the standard deviation.

(4) It is simple to understand and easy to compute.

Disadvantages of Mean Deviation :

(1) Mean deviation ignores the algebraic signs of deviations and as such it is not capable of further algebraic treatment.

(2) It is not an accurate measure, particularly when it is calculated from mode.

(3) It is not popular as standard deviation.

Uses of Mean Deviation :

Because of simplicity in computation, it has drawn the attention of economists and businessmen. It is useful reports meant for public.

5.3.2.3. Standard Deviation :

In calculating mean deviation we ignored the algebraic signs, which is mathematically illogical. This drawback is removed in calculating standard deviation, usually denoted by 'σ' (read as sigma)

Definition : Standard deviation is the square root of the arithmetic average of the squares of all the deviations from the mean. In short, it may be defined as root-mean-square deviation from the mean.

If x is the mean of $x_1, x_2, \ldots\ldots, x_n$, then σ is defined by

$$\sqrt{\left[\frac{1}{n}\left\{(x_1-\bar{x})^2+\ldots\ldots+(x_n-\bar{x})^2\right\}\right]} = \sqrt{\frac{1}{n}\sum(x_i-\bar{x})^2}$$

Different formulae for computing s. d.

(a) For simple observations or variates.

If be A.M. of $x_1, x_2 \ldots\ldots\ldots, x_n$, then $\sigma = \sqrt{\frac{1}{n}\sum(x_i-\bar{x})^2}$

(b) For simple or group frequency distribution

For the variates $x_1, x_2, x_3, \ldots\ldots, x_n$, if corresponding frequencies are $f_1, f_2, f_3, \ldots\ldots, f_n$

Then $\sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}}$ .... (2) Where, $\bar{x}$ = weighted A. M.

Note : If variates are all equal (say K), then $\sigma = 0$, as $\bar{x}$ = K and $\sum(x - \bar{x}) = 0$

Example 49: For observations 4, 4, 4, 4, s = 0 as $\bar{x} = 4$ and $\sum(4-4) = 0$

Short cut method for calculating s.d.

If $\bar{x}$ (A. M.) is not an integer, in case (1), (2) ; then the calculation is lengthy and time consuming. In such case, we shall follow the following formulae for finding s.d.

(c) For simple observations, $\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$ .... (3)

Where, d = x – A, A is assumed mean.

(d) For simple (or group) frequency distribution

$$\sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}, \quad \text{Where, } d = x - A$$

(e) For group frequency distribution having equal class interval

$$\sigma = \sqrt{\frac{\sum fd'^2}{\sum f} - \left(\frac{\sum fd'}{\sum f}\right)^2} \times i \quad .... (5) \quad \text{where, } d' = \frac{x - A}{i}$$

(This is known as step deviation method)

Observation $\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}} = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$ .... (6)

(The proof is not shown at present)

Note : Formula (3) may be written as, for step deviation where $d' = \frac{x - A}{i}$

$$\sigma = \sqrt{\frac{\sum d'^2}{n} - \left(\frac{\sum d'}{n}\right)^2} \times i \qquad .... (7)$$

5.3.2.3.1. Computation for Standard Deviation :

(A) For individual observations computation may be done in two ways :

(a) by taking deviations from actual mean. Steps to follow—

(1) Find the actual mean, i.e. $\bar{x}$.

(2) Find the deviations from the mean, i.e., d.

(3) Make squares of the deviations, and add up, i.e. $\sum d^2$.

(4) Divide the addition by total number of items, i.e., find d^2 / n and hence make square root of it.

(b) by taking deviations from assumed mean. Steps to follow—

(1) Find the deviations of the items from an assumed mean and denote it by d find also $\sum d$.

(2) Square the deviations, find $\sum d^2$.

(3) Apply the following formula to find standard deviation.

$$S.D.(\sigma) = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$

Example 50 : Find s.d. of (₹) 7, 9, 16, 24, 26. Calculation of s.d. by methods (a) Taking deviations of Sum (b) Taking deviations from Assumed Mean

Method (a) : Taking deviation from A.M.			Method (b) : Taking deviation from assumed mean		
Variate (₹)	A.M. (16.4)		Variate (₹)	A.M. (16)	
x	d	d^2	x	d	d^2
7	–9.4	88.36	7	–9	81
9	–7.4	54.76	9	–7	49
16	–0.4	0.16	16	0	0
24	7.6	57.76	24	8	64
26	9.6	92.16	26	10	100
Total	––	293.20	–	2	294

For method (a) : $\bar{x}\,(A.M.) = \frac{82}{5} = 16.40$

$$\sigma\,(= s.d) = \sqrt{\frac{1}{n}\sum(x - \bar{x})^2} = \sqrt{\frac{1}{n}\sum d^2} = \sqrt{\frac{1}{5} \times 293.20} = \sqrt{58.64} = ₹\ 7.66$$

Here the average or A.M. 16.40 and the variates deviate on an average from the A.M. by ₹ 7.66.

For method (b) : Let A (assumed mean) = 16

$$\sigma(= \text{s.d.}) = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}\text{, by using formula (3)}$$

$$= \sqrt{\frac{294}{5} - \left(\frac{2}{5}\right)^2} = \sqrt{58.8 - (0.4)^2} = \sqrt{58.8 - 0.16} = ₹\ 7.66.$$

Note : If the actual mean is in fraction, then it is better to take deviations from an assumed mean, for avoiding too much calculations.

(B) For discrete series (or Simple Frequency Distribution). There are three methods, given below for computing Standard Deviation.

(a) Actual Mean, (b) Assumed Mean, (c) Step Deviation.

For (a) the following formula are used.

This method is used rarely because if the actual mean is in fractions, calculations take much time.

$$\sigma = \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \text{ or } \sqrt{\frac{\sum fd^2}{\sum f}}\ ;\ d = x - \bar{x}$$

(In general, application of this formula is less)

For (b), the following steps are to be used :–

(i) Find the deviations (from assumed mean), denote it by d.

(ii) Obtain $\sum fd$.

(iii) Find $\sum fd^2$, i.e. (fd × d and then take $\sum$, and hence use the formula.

$$= \sqrt{\frac{fd^2}{f} - \frac{fd}{f}^{\,2}}$$

Example 51 : Find the Standard deviation of the following series :

x	f
10	3
11	12
12	18
13	12
14	3
Total	48

Solution :

Table : Calculation of Standard Deviation

X	f	Devn. From Ass. Mean (12) d	fd	d^2	fd^2
(1)	(2)	(3)	(4) = (2) × (3)	(5) = (3) × (3)	(6) (2) × (5)
10	3	–2	–6	4	12
11	12	–1	–12	1	12
12	18	0	0	0	0
13	12	1	12	1	12
14	3	2	6	4	12
Total	48		0		48

$$\sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} = \sqrt{\frac{48}{48} - \frac{0}{48}} = \sqrt{1} = 1$$

For (c) the following formula is used.

The idea will be clear from the example shown below :

Formula is, $\sigma = \sqrt{\frac{fd'^2}{f} - \frac{fd'}{f}^{\;2}} \times i$ where d′ = step deviation, i = common factor.

Example 52: Find the standard deviation for the following distribution :

x	f
4.5	2
14.5	3
24.5	5
34.5	17
44.5	12
54.5	7
64.5	4

Solution :

Table : Calculation of Standard Deviation

x	f	d	$d' = \frac{d}{10}$	fd¢	fd¢^{2}
4.5	2	–30	–3	–6	18
14.5	3	–20	–2	–6	12
24.5	5	–10	–1	–5	5
34.5	17	0	0	0	0
44.5	12	10	1	12	12
54.5	7	20	2	14	28
64.5	4	30	3	12	36
	$\sum f = 50$	–	–	$\sum fd' = 21$	$\sum fd'^2 = 111$

$$\sigma = \sqrt{\left\{\frac{\sum fd'^2}{\sum f} - \left(\frac{\sum fd'}{\sum f}\right)^2\right\}} \times i = \sqrt{\left\{\frac{111}{50} - \left(\frac{21}{50}\right)^2\right\}} \times 10$$

$$= \sqrt{(2.22 - 0.1764)} \times 10 = 1.4295 \times 10 = 14.295.$$

(C) For Continuous Series (or group distribution) : Any method discussed above (for discrete series) can be used in this case. Of course, step deviation method is convenient to use. From the following example, procedure of calculation will be clear.

Example 53 : Find the standard deviation from the following frequency distribution.

Weight (kg.)	No. of persons
44–46	3
46–48	24
48–50	27
50–52	21
52—54	5
Total	80

Solution:

Table : Calculation of Standard deviation

Weight (kg.)	mid. pt. x	Frequency f	devn. $(d = x - 49)$	$d' = \frac{d}{2}$	fd'	fd'^2
44–46	45	3	–4	–2	–6	12
46–48	47	24	–2	–1	–24	24
48–50	49	27	0	0	0	0
50–52	51	21	2	1	21	21
52–54	53	5	4	2	10	20
Total	–	80	–	–	1	77

Let A (assumed mean) = 49

$$\sigma = \sqrt{\left\{\frac{\sum fd'^2}{\sum f} - \left(\frac{\sum fd'}{\sum f}\right)^2\right\}} \times i = \sqrt{\frac{77}{80} - \left(\frac{1}{80}\right)^2} \times 2$$

$$= \sqrt{\left(0.9625 - \frac{1}{6400}\right)} \times 2$$

$$= \sqrt{0.9625 - 0.00016} \times 2 = \sqrt{0.96234} \times 2 = 0.9809 \times 2 = 1.96 \text{ kg.}$$

5.3.2.3.2. MATHEMATICAL PROPERTIES OF STANDARD DEVIATION :

Combined Standard Deviation.

We can also calculate the combined standard deviation for two or more groups, similar to mean of composite group. The required formula is as follows :

$$\sigma_{12} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2}{n_1 + n_2}}$$

where σ_{12} = combined standard deviation of two groups.

σ_1 = standard deviation of 1st group.

σ_2 = standard deviation of 2nd group.

$$d_1 = \bar{x}_1 - \bar{x}_{12};\ d_2 = \bar{x}_2 - \bar{x}_{12},\ \text{where } \bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

For Three Groups

$$\sigma_{123} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_3\sigma_3^2 + n_1d_1^2 + n_2d_2^2 + n_3d_3^2}{n_1 + n_2 + n_3}}$$

Where $d_1 = \bar{x}_1 - \bar{x}_{123}; d_2 = \bar{x}_2 - \bar{x}_{123}; d_3 = \bar{x}_3 - \bar{x}_{123}$

Example 54 : Two samples of sizes 40 and 50 respectively have the same mean 53, but different standard deviations 19 and 8 respectively. Find the Standard Deviations of the combined sample of size 90.

Solution:

Here, $n_1 = 40, \bar{x}_1 = 53, \sigma_1 = 19;\ n_2 = 50, \bar{x}_2 = 53, \sigma_2 = 8$

Now, $\bar{x}_{12} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \dfrac{40 \times 53 + 50 \times 53}{40 + 50} = \dfrac{2120 + 2650}{90} = \dfrac{4770}{90} = 53$

Now, $d_1 = \bar{x}_1 - \bar{x}_{12} = 53 - 53 = 0,\ d_2 = \bar{x}_2 - \bar{x}_{12} = 53 - 53 = 0,$

$$\sigma_{12} = \sqrt{\left\{\frac{40(19)^2 + 50(8)^2 + 40(0)^2 + 50(0)^2}{40 + 50}\right\}}$$

$$= \sqrt{\left(\frac{14440 + 3200}{90}\right)} = \sqrt{\frac{17640}{90}} = \sqrt{196} = 14$$

VARIANCE :

The square of the Standard Deviation is known as Variance.

COEFFICIENT OF VARIATION :

It is the ratio of the Standard Deviation to the Mean expressed as percentage. This relative measure was first suggested by Professor Kari Pearson. According to him, coefficient is the percentage variation in the Mean, while Standard Deviation is the total variation in the Mean.

Symbolically,

Coefficient of variation (V) $= \dfrac{\sigma}{\bar{x}} \times 100 =$ Coefficient of stand. deviation × 100.

Note : The coefficient of variation is also known as coefficient at variability. It is expressed as percentage.

Example 55 : If Mean and Standard deviation of a series are respectively 40 and 10, then the coefficient of variations would be 10 / 40 × 100 = 25%, which means the standard deviation is 25% of the mean.

Example 56: An analysis of the monthly wages paid to workers in two firms, A and B, belonging to the same industry gives the following results :

	Firm A	Firm B
No. of wage-earners	586	648
Average monthly wages	₹ 52.5	₹ 47.5
Variance of distribution of wages	100	121

(a) Which firm A and B pays out the largest amount as monthly wages?

(b) Which firm A and B has greater variability in individual wages?

(c) Find the average monthly wages and the standard deviation of the wages of all the workers in two firms A and B together.

Solution :

(a) For firm A : total wages = 586 × 52.5 = ₹ 30,765.

For firm B : Total wages = 648 × 47.5 = ₹ 30,780. i.e. Firm B pays largest amount.

(b) For firm A : $\sigma^2 = 100 \quad \therefore \quad \sigma = 10$

Now, $V = \frac{\sigma}{\text{Mean}} \times 100 = \frac{10}{52.5} \times 100 = 19.04$

For firm B : $\sigma^2 = 121 \qquad \therefore \sigma = 11$

$V = \frac{11}{47.5} \times 100 = 23.16$

∴ Firm B has greater variability, as its coefficient of variation is greater than that of Firm A.

(c) Here, $n_1 = 586$, $\bar{x}_1 = 52.5$, $\sigma_1 = 10$

$n_2 = 648$, $\bar{x}_2 = 47.5$ $\sigma_2 = 11$

$$\therefore \bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{586 \times 52.5 + 648 \times 47.5}{586 + 648} = \frac{30{,}765 + 30{,}780}{1234}$$

$$= \frac{61{,}545}{1{,}234} = 49.87 = ₹\ 49.9$$

Again, $d_1 = \bar{x}_1 - \bar{x}_2 = 52.5 - 49.9 = 2.6$; $d_2 = 47.5 - 49.9 = -2.4$

$$\therefore \sigma_{12} = \sqrt{\left\{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2}{n_1 + n_2}\right\}}$$

$$= \sqrt{\left\{\frac{586(10)^2 + 648(11)^2 + 586(2.6)^2 + 648(-2.4)^2}{586 + 648}\right\}}$$

$$= \sqrt{\left\{\frac{58600 + 78408 + 3962 + 3733}{1234}\right\}}$$

$$= \sqrt{\frac{144703}{1234}} = 10.83 \quad \text{(Calculation by log table)}$$

Example 57 : In an examination a candidate scores the following percentage of marks :

English	2nd language	mathematics	Science	Economics
62	74	58	61	44

Find the candidates weighted mean percentage weighted of 3, 4, 4, 5 and 2 respectively are allotted of the subject. Find also the coefficient of variation.

Solution:

Table : Calculation of Coefficient of Variation

Marks	f	fx	d = x – 61	fd	fd^2
62	3	186	1	3	3
74	4	296	13	52	676
58	4	232	–3	–12	36
61	5	305	0	0	0
44	2	88	–17	–34	578
Total	18	1107		9	1293

Weighted mean percentage $= \frac{\sum fx}{\sum f} = \frac{1107}{18}$ = 61.5 marks

s.d. (s) $= \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} = \sqrt{\frac{1293}{18} - \left(\frac{9}{18}\right)^2} = \sqrt{71.83 - 0.25}$

$= \sqrt{71.58} = 8.46$

Coeff. of variation $= \frac{\text{s.d}}{\text{A.M.}} \times 100 = \frac{8.46}{61.5} \times 100 = 13.76\%.$

Example 58 : The A.M. of the following frequency distribution is 1.46. Find f_1 and f_2.

No. of accidents :	0	1	2	3	4	5	total
No. of days :	46	f_1	f_2	25	10	5	200

Also find coefficient of variation.

Solution:

Putting these values of f_1 and f_2 we find the following distribution :

Table : Calculation of Coefficient of Variation

x	f	d	fd	fd^2
0	46	–2	–92	184
1	76	–1	–76	76
2	38	0	0	0
3	25	1	25	25
4	10	2	20	40
5	5	3	15	45
Total	200	–	–108	370

$$AM(\bar{x}) = 2 + \frac{-108}{200} = 2 - 0.54 = 1.46$$

$$\sigma = \sqrt{\frac{370}{200} - \left(\frac{-108}{200}\right)^2} = \sqrt{1.85 - (0.54)^2}$$

$$= \sqrt{1.85 - 0.2916} = \sqrt{1.5584}$$

= 1.248 = 1.25 (app.)

Now coeff. of variation $= \frac{1.25}{1.46} \times 100 = 85.62\%$ (app.)

Advantages of Standard Deviation :

1. Standard deviation is based on all the observations and is rigidly defined.
2. It is amenable to algebraic treatment and possesses many mathematical properties.
3. It is less affected by fluctuations of sampling than most other measures of dispersion.
4. For comparing variability of two or more series, coefficient of variation is considered as most appropriate and this is based on standard deviation and mean.

Disadvantages of Standard Deviation :

1. It is not easy to understand and calculate.
2. It gives more weight to the extremes and less to the items nearer to the mean, since the squares of the deviations of bigger sizes would be proportionately greater than that which are comparatively small. The deviations 2 and 6 are in the ratio of 1 : 3 but their squares 4 and 36 would be in the ratio of 1 : 9.

Uses of Standard Deviation :

It is best measure of dispersion, and should be used wherever possible.

5.4 COEFFICIENT QUARTILE & COEFFICIENT VARIATION

Example 59:

Calculate co-efficient of quartile deviation and co-efficient of variation from the following data :

Marks	No. of student
Below 20	8
" 40	20
" 60	50
" 80	70
" 100	80

Solution :

Table : Calculation of Coefficient of Quartile Deviation and Cofficient of Variation

Marks	m.p.n	No. of students	m-50 20 (d)	fd	fd^2	cf
0-20	10	8	-2	-16	32	8
20-40	30	12	-1	-12	12	20
40-60	50	30	0	0	0	50
60-80	70	20	+1	+20	20	70
80-100	90	10	+2	+20	40	80
		n=80		fd = 12	$fd^2 = 104$	

Mean :

$$X = A + \frac{\Sigma fd}{N} \times C$$

$$= 50 + \frac{12}{80} \times 20$$

= 50 + 3 = 53

Standard deviation

$$= \sqrt{\frac{\Sigma fd^2}{N} - \frac{\Sigma fd}{N}^2} \times C$$

$$= \sqrt{\frac{104}{80} - \frac{12}{80}^2} \times 20$$

$$= \sqrt{1.3 - (0.15)^2} \times 20$$

$$= \sqrt{1.3 - 0.0225} \times 20$$

$$= \sqrt{1.2775} \times 20 = 1.13 \times 20 = 22.6$$

$$C.V._{\cdot 2} = \frac{\sigma}{\lambda} \times 100$$

$$= \frac{22.6}{53} \times 100$$

= 42.64%

$$\text{Co-efficient of Q.D.} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

Q_1 = Size of N/4th item

= Size of 80/4 = 20th item

Q_1 lies in the class 20 - 40

$$Q_1 = L + \frac{\frac{N}{4} - cf}{f} \times i$$

$$= 20 + \frac{20-8}{12} \times 20$$

$= 20 + 20 = 40$

Q_3 = Size of $\frac{3N}{4}$ th item

= Size of $\frac{3 \times 80}{4}$ = 60th item

Q_3 lies in the class 60 - 80

$$Q_3 = L_1 + \frac{\frac{3N}{4} - cf}{f} \times i$$

$$= 60 + \frac{60-50}{20} \times 20$$

$= 70$

Co-efficient of Q.D $= \frac{70-40}{70+40} = \frac{30}{110} = 0.27$

SELF EXAMINATION QUESTIONS

(A) Regarding Range.

1. Daily wages in ₹ of 7 workers are as follows :

(₹) : 12, 8, 9, 10, 7, 14, 15. Calculate range. [Ans. ₹ 8]

2. Find range :

Weight (kg) : 40, 51, 47, 39, 60, 48, 64, 61, 57. [Ans. 25 kg]

3. The marks obtained by 6 students are 24, 12, 16, 11, 40, 42. Find range. If now the highest mark is omitted, find the percentage change in range. [Ans. 31 marks, 6.45]

(B) Regarding MEAN DEVIATION :

4. Find Mean Deviation about mean of the numbers given :

(i) 31, 35, 29, 63, 55, 72, 37. [Ans. 14.9]

(ii) 29, 35, 51, 63, 78, 106, 128 [Ans. 29.143]

5. Find M.D. about median of :

13, 84, 68, 24, 96, 84, 27. [Ans. 32.286]

6. Find M.D. about A.M. of the table given below :

x	f
2	1
4	4
6	6
8	4
10	1

Find also coefficient of mean dispersion. [Ans. 2.3125, 0.420]

7. From the following table find coefficient of mean dispersion about :

(i) A.M., (ii) Median.

Marks	Frequency
10	8
15	12
20	15
30	10
40	3
50	2

[Ans. (i) 0.363, (ii) 0.36]

8. From the following frequency distribution find M.D. about median :

C.I.	f
2–4	3
4–6	4
6–8	2
8–10	1

[Ans. 1.4]

9. Find M.D. about A.M. of the table :–

Weight (lb)	Students
95–105	20
105–115	26
115–125	38
125–135	16

[Ans. 8.6 lbs]

(C) Regarding STANDARD DEVIATION :

10. Calculate standard deviation of the following numbers :

(i) 9, 7, 5, 11, 3 [Ans. 2.83]

(ii) 1, 2, 3, 4, 5 [Ans. 1,414]

(iii) 1, 2, 3, 4, ….. 9, 10 [Ans. 2.87]

(iv) 4, 5, 6, 6, 7, 8 [Ans. 1.29]

(v) 9, 7, 5, 11, 1, 5, 7, 3. [Ans. 3.072]

11. The frequency distribution of heights of 50 persons is shown below :

Height (inches)	No. of persons
62	8
64	13
66	17
68	12

Find s.d. and variance. [Ans. 2.02 inch. 4.50 sq. inch]

12. Find s.d. from the tables :

(i)

Age (yrs.)	Persons
30	64
40	132
50	153
60	140
70	51

[Ans. 11.64 yrs.]

(ii)

Class-limits	Frequency
4.5	1
14.5	5
24.5	12
34.5	22
44.5	17
54.5	9
64.5	4

[Ans. 13.25]

13. Compute s.d. from the following tables :

(i)

Height (inch)	Students
60–62	34
62–64	27
64–66	20
66–68	13
68–70	5
Total	100

[Ans. 2.41 inch]

(ii)

Marks	Students
0–10	5
10–20	8
20–30	15
30–40	16
40–50	6

[Ans. 10.77 marks]

14. Find the coefficient of variation of numbers : 1, 2, 3, 4, 5. [Ans. 47.13%]

15.

Marks	Students
10	8
20	12
30	20
40	10
50	7
60	3

Find coefficient of variation. [Ans. 64.81%]

16. Run-scores in 10 innings of two cricketers are as follows :

A	B
31	19
28	31
47	48
63	53
71	67
39	90
10	10
60	62
96	40
14	80

Find which batsman is more consistent in scoring. [Ans. Batsman B]

17. The A.M.'s of two samples of sizes of 60 are 90are respectively 52 and 48, the s.d. are 9 and 12. Obtain the mean and s.d. of the sample of size 150 obtained by combining the two samples.

[Ans. 49.6, 11.1]

18. The first of two samples has 100 items with mean 15 and s.d. 3. If the whole group has 250 items with mean 15.6 and s.d. $\sqrt{13.44}$, find the s.d. of the second group. [Ans. 4]

OBJECTIVE QUESTIONS :

1. Find the range of 6, 18, 17, 15, 14 [Ans. 12]
2. Find Mean Deviation (M.D.) of 4, 8, 12 (cm) about A.M. [Ans. 2.67 cm]
3. Find M.D. about median of 4, 8, 10 (kg) [Ans. 2 kg.]
4. Find S.D. of (i) 2, 5, 8 (ii) 2, 6 [Ans. (i) $\sqrt{6}$ (ii) 2]
5. Find variance of 2, 5, 8 [Ans. 6]
6. Find S.D. from the given data : n = 10, $\Sigma x = 40$, $\Sigma x^2 = 250$ [Ans. 3]
7. If n = 10, $\Sigma x = 120$, $\Sigma x^2 = 1690$; find s.d. [Ans. 5]
8. If variance = 16, A.M. = 50, find coefficient of variation [Ans. 8%]
9. Find variance of x, if it's A.M. is 6 and coefficient of variation is 50%. [Ans. 9]
10. Find mean, if c.v. = 5% and variance = 4 [Ans. 40]
11. Coefficient of variation of a distribution is 25%, it it means what?

    [Ans. s.d. is 25% of A.M.]
12. If each term of variates is increased by 2, what will be the effect on (i) A.M. (ii) range and (iii) s.d.

    [Ans. (i) increased by 2, (ii) & (iii) no change]
13. If each item is doubled what will be effect on

    (i) A.M. (ii) Range (iii) s.d. [Ans. (i), (ii) & (iii) doubled]
14. Two variables x and y are related by y = 4x – 7. If s.d. of x is 2, find s.d. of y [Ans. 8]
15. Two variates x and y are given by y = 2 – 3x, s.d. of x is 2, find s.d. of y [Ans. 6]
16. Compute s.d. of 6 numbers 7, 7, 7, 9, 9, 9. [Ans. 1]
17. Compute M.D. of 6 numbers 4, 4, 4, 6, 6, 6 [Ans. 1]
18. Means and S.D. of runs of 10 innings of two players are as follows :

    First player : mean = 50, s.d. = 4

    Second player, mean = 40, s.d. = 5

    Find who is more consistent in scoring runs?

    [Ans. First player]

19. If $2x_i + 3y_i = 5$ for I = 1, 2, ..., n and mean deviation of $x_1, x_2, \ldots, x_n$ about their mean is 12, find the mean deviation of $y_1, y_2, \ldots, y_n$ about their mean.

[Ans. $\frac{19}{3}$]

20. If the means of two groups of 30 and 50 observation are equal and their standard deviation are 8 and 4 respectively, find the grouped variance. [Ans. 5.83]

21. For 10 values $x_1, x_2, \ldots, x_{10}$ of a variable x, $\sum_{i=1}^{10} x_i = 110$, and $\sum_{i=1}^{10} (x_i - 5)^2 = 1000$, find variance of x [Ans. 64]

22. If the relations between two variables x and y be $2x - y + 3 = 0$ and range of x be 10, then find the range of y. [Ans. 20]

23. Runs made by two groups G_1 and G_2 of cricketers have means 50 and 40 and variance 49 and 36 respectively. Find which group is more constant in scoring runs. [Ans. G_1]

24. If A.M. and coefficient of variation of a variable x are 10 and 50% respectively, find the variance of x. [Ans. 25]

Study Note - 6

CORRELATION AND REGRESSION

This Study Note includes

6.1 CORRELATION & CO-EFFICIENT

Till the previous chapter we have been mainly concerned with univariate data. In this chapter we study bivariate and multivariate populations.

According to Ya-lun Chou, "There are two related but distinct aspects of the study of association between variables. Correlation analysis and regression analysis. Correlation analysis has the objective of determining the degree or strength of the relationship between variables. Regression analysis attempts to establish the nature of the relationship between variables – that is, to study the functional relationship between the variables and thereby provide a mechanism of prediction, or forecasting."

6.1.1. Meaning

In our daily lives we notice that the bigger the house the higher are its upkeep charges, the higher rate of interest the greater is the amount of saving, the rise in prices bring about a decrease in demand and the devaluation of country's currency makes export cheaper or import dearer.

The above example clearly shows that there exists some kind of relationship between the two variables.

Croxton and Cowden rightly said, "when relationship between two variables is of quantitative nature the appropriate statistical tool for measuring and expressing it in formula is known as correlation. Thus correlation is a statistical device which helps in analyzing the relationship and also the covariation of two or more variables.

According to Simpson and Kafta "correlation analysis deals with the association between two or more variables." If two variables vary in such a way that movements in one are accompanied by movements in the other, then these quantities are said to be correlated.

6.1.2. Importance of Correlation

A car owner knows that there is a definite relationship between petrol consumed and distance travelled. Thus on the basis of this relationship the car owner can predict the value of one on the basis of other. Similarly if he finds that there is some distortions of relationship, he can set it right.

Correlation helps in the following ways

1. It helps to predict event and the events in which there is time gap *i.e.* it helps in planning
2. It helps in controlling events.

6.1.3. Types of Correlation

Correlation can be classified under the following heads-

1. Positive and negative correlation
2. Simple multiple and partial correlation
3. Linear and non-linear correlation

6.1.4. Positive and Negative Correlation

Two variables are said to be positively correlated when both the variables move in the same direction. The correlation is said to be positive (directly related) when the increase in the value of one variable is accompanied by an increase in the value of the other variable and *vice versa.*

Two variables are said to be negatively correlated when both the variables move in the opposite direction. The correlation is said to be negative (inversely related) when the increase in the value of one variable is accompanied by a decrease in the value of the other variable and *vice versa.*

6.1.5. Simple, Multiple and Partial Correlation

Correlation is said to be simple when only two variables are studied.
In multiple correlation three or more variables are studied simultaneously.

In partial correlation though more than two variables are recognised, but only two are considered to be influencing each other; and the effect of other influencing variables are kept constant.

6.1.6. Linear and Non-linear Correlation

If the amount of change in one variable tends to bear a constant ratio to the amount of change in the other variable, then the correlation is said to be linear.

The correlation is said to be non-linear if the amount of change in one variable does not bear a constant ratio to the amount of change in the other related variable.

6.1.7. Measurement of Correlation

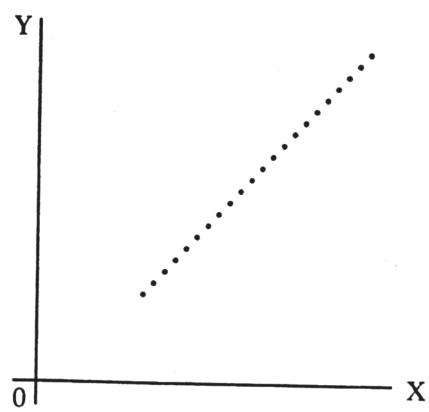
The correlation can be measured by any of the following methods-

1. Scatter Diagram
2. Karl Pearson's coefficient of correlation
3. Rank correlation coefficient

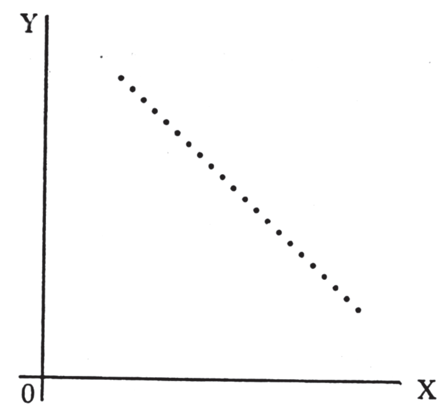
6.1.8. Scatter Diagram Method

The scatter diagram represents graphically the relation between two variables Xand Y. For each pair of Xand Y, one dot is put and we get as many points on the graph as the number of observations. Degree of correlation between the variables can be estimated by examining the shape of the plotted dots.

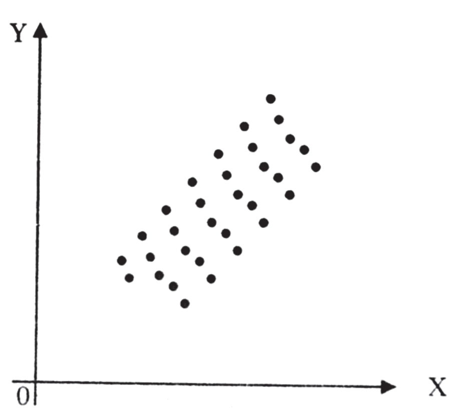
Following are some scattered diagrams showing varied degrees of correlation.



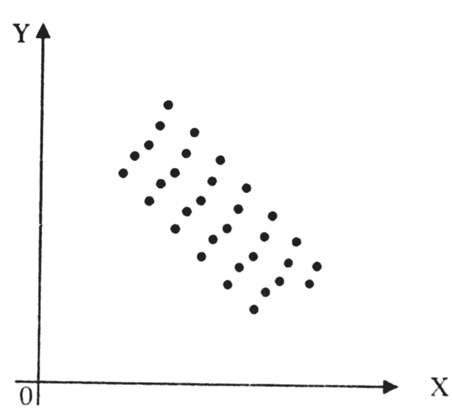
Perfect positive correlation; $r = 1$



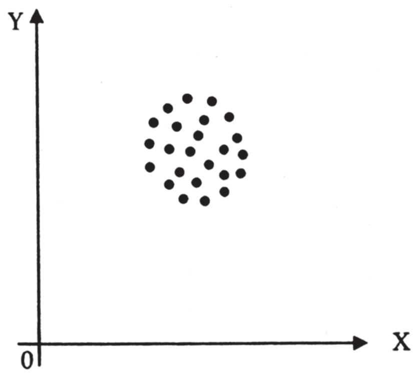
Perfect negative correlation; $r = -1$



Low degree of positive correlation; $r > 1$



Low degree of negative correlation; $r < 1$



No correlation; r— 0

Advantages

(1) It is very easy to draw a scatter diagram

(2) It is easily understood and interpreted

(3) Extreme items does not unduly affect the result as such points remain isolated in the diagram

Disadvantages

(1) It does not give precise degree of correlation

(2) It is not amenable to further mathematical treatment

6.1.9. Karl Pearson's Coefficient of Correlation

The measure of degree of relationship between two variables is called the correlation coefficient. It is denoted by symbol r. The assumptions that constitute a bivariate linear correlation population model, for which correlation is to be calculated, includes the following-(ya-lun chou)

1. Both X and Y are random variables. Either variable can be designated as the independent variable, and the other variable is the dependent variable.
2. The bivariate population is normal. A bivariate normal population is, among other things, one in which both X and Y are normally distributed.
3. The relationship between X and Y is, in a sense, linear. This assumption implies that all the means of Y's associated with X values, fall on a straight line, which is the regression line of Y on X. And all the means of X's associated with Y values, fall on a straight line, which is the regression line of X on Y. Furthermore, the population regression lines in the two equations are the same if and only if the relationship between Y and X is perfect- that is $r = \pm 1$. Otherwise, with Y dependent, intercepts and slopes will differ from the regression equation with X dependent.

This method is most widely used in practice. It is denoted by symbol V. The formula for computing coefficient of correlation can take various alternative forms depending upon the choice of the user.

METHOD I — WHEN DEVIATIONS ARE TAKEN FROM ACTUAL ARITHMETIC MEAN

(A) WHEN STANDARD DEVIATIONS ARE GIVEN IN THE QUESTION.

$$r = \frac{xy}{N(\sigma_x \sigma_y)}$$

Where x = Deviations taken from actual mean of X series

Y = Deviations taken from actual mean of Y series

N = Number of items

σ_x = Standard deviation of X series

σ_y = Standard deviation of Y series

(B) WHEN STANDARD DEVIATIONS ARE NOT GIVEN IN THE QUESTION

$$r = \frac{\sum xy}{\sqrt{\sum x^2 \times \sum y^2}}$$

Where Σxy = Sum of product of deviations of X and Y series from actual mean

Σx^2 = Sum of squares of deviation of X series from its mean

Σy^2 = Sum of squares of deviation of Y series from its mean

Example : 1

Find correlation between marks obtained by 10 students in mathematics and statistics

X	2	4	6	6	8	9	10	4	7	4
Y	12	12	16	15	18	19	19	14	15	10

Solution :

Calculation of coefficient of correlation

X	***Y***	***X***	***y***	***X²***	***y²***	***xy***
2	12	-4	-3	16	**9**	12
4	12	-2	-3	4	**9**	6
6	16	0	1	0	1	0
6	15	0	0	0	0	0
8	18	2	3	4	9	6
9	19	3	4	9	16	12
10	19	4	4	16	16	16
4	14	-2	-1	4	1	2
7	15	1	0	1	0	0
4	10	-2	-5	4	25	10
$\Sigma X = 60$	$\Sigma Y = 150$	$\Sigma x = 0$	$\Sigma y = 0$	$\Sigma x^2 = 58$	$\Sigma y^2 = 86$	$\Sigma xy = 64$

Calculation by Method 1(a)

$$\overline{X} = \frac{X}{N} = \frac{60}{10} = 6$$

$$\overline{Y} = \frac{Y}{N} = \frac{150}{10} = 15$$

$$r = \frac{xy}{N\sigma_X \sigma_y}$$

$$\sigma_x = \sqrt{\frac{x^2}{N}} = \sqrt{\frac{58}{10}}$$

$$= \sqrt{5.8} = 2.408$$

$$\sigma_y = \sqrt{\frac{y^2}{N}} = \sqrt{\frac{86}{10}} = 2.93$$

$$r = \frac{64}{10 \times 2.408 \times 2.932}$$

$$= \frac{64}{70.602}$$

$$r = 0.906$$

*(**Note :** The above method should be used when specifically asked for, or if standard deviations are already given in the question, otherwise the following method should be used as it is less cumbersome)*

Calculation by Method 1*(b)*

$$r = \frac{xy}{\sqrt{x^2 \quad y^2}}$$

$$= \frac{64}{\sqrt{58 \times 86}}$$

$$= \frac{64}{\sqrt{4988}}$$

$$= \frac{64}{70.62}$$

$$r = 0.906$$

METHOD II WHEN DEVIATIONS ARE TAKEN FROM ASSUMED MEAN

This method is generally used when actual mean of X series or of Y series or both are in decimals, in which case using method I becomes tedious; in such a case deviations are taken from assumed mean to simplify the calculations.

$$r = \frac{d_x d_y - \frac{(\; d_x)(\; d_y)}{N}}{\sqrt{d_x^2 - \frac{(\; d_x)^2}{N}}\sqrt{d_y^2 - \frac{(\; d_y)^2}{N}}}$$

Where

d_x = Deviations taken from assumed mean of X series = $(X - A_x)$

d_y = Deviations taken from assumed mean of Y series = $(X - A_y)$

Σd_x = Sum of deviations of X series from its assumed mean

Σd_y = Sum of deviations of Y series from its assumed mean

Σd_x^2 = Sum of squares of deviations of X series from its assumed mean

Σd_y^2 = Sum of squares of deviations of Y series from its assumed mean

$\Sigma d_x d_y$ = Sum of the product of deviations of X and Y series from an assumed mean

N = number of observations

r = Correlation coefficient

Example 2 :

Calculate coefficient of correlation from following data

X	0	15	15	14	10	12	10	8	16	15
Y	20	15	12	10	8	5	6	15	12	18

Solution :

Table : Calculation of coefficient of correlation

X	***Y***	$d_x = x - A_x$	$d_y = y - A_y$	d_x^2	d_y^2	$d_x d_y$
0	20	-10	5	100	25	-50
15	15	5	0	25	0	0
15	12	5	-3	25	9	-15
14	10	4	-5	16	25	-20
10	8	0	-7	0	49	0
12	5	2	-10	4	100	-20
10	6	0	-9	0	81	0
8	15	-2	0	4	0	0
16	12	6	-3	36	9	-18
15	18	5	3	25	9	15
ΣX=115	ΣY= 121	Σd_x = 15	Σd_x = -29	Σd^2 = 235	Σd^2 = 307	$\Sigma d_x d_y$= -108

Since mean of X and Y are in decimals *i.e.* 11.5 and 12.1 respectively hence we would solve by method II

$$r = \frac{d_x d_y - \frac{(d_x)(d_y)}{N}}{\sqrt{d_x^2 - \frac{(d_x)^2}{N}}\sqrt{d_y^2 - \frac{(d_y)^2}{N}}}$$

Taking A_x as 10 and A_y as 15

$$r = \frac{-108 - \frac{(15 \times -29)}{10}}{\sqrt{235 - \frac{(15)^2}{10}}\sqrt{307 - \frac{(-29)^2}{10}}}$$

$$r = \frac{-64.5}{\sqrt{212.5 \times 222.9}}$$

$$= \frac{-64.5}{\sqrt{47366.25}} = \frac{-64.5}{217.63}$$

$= -0.296$

Example 3 :

Find correlation between age of husband and age of wife.

Age of Husband (X)	46	54	56	56	58	60	62
Age of Wife (Y)	36	40	44	54	42	58	54

Solution :

Table : Calculation of correlation coefficient

X	Y	$d_x = x - A_x$	$d_y = y - A_y$	d_x^2	d_y^2	$d_x d_y$
46	36	-10	-9	100	81	90
54	40	-2	-5	4	25	10
56	44	0	-1	0	1	0
56	54	0	9	0	81	0
58	42	2	-3	4	9	-6
60	58	4	13	16	169	52
62	54	6	9	36	81	54
ΣX=392	ΣY=328	Σd_x = 0	Σd_y = 13	Σd_x^2=160	Σd_y^2=447	$\Sigma d_x d_y$=200

Since mean of Y is in decimals i.e. 46.85, we would solve it by short cut method

$$r = \frac{d_x d_y - \frac{(d_x)(d_y)}{N}}{\sqrt{d_x^2 - \frac{(d_x)^2}{N}}\sqrt{d_y^2 - \frac{(d_y)^2}{N}}}$$

Taking A_x as 10 and A_y as 15

N = 7

$$r = \frac{200 - \left(\frac{0x13}{7}\right)}{\sqrt{160 - \frac{(0)^2}{10}}\sqrt{447 - \frac{(13)^2}{10}}}$$

$$= \frac{200}{\sqrt{160} x \sqrt{422.8}}$$

$$= \frac{200}{\sqrt{67648}}$$

$$= \frac{200}{260.09}$$

r = 0.768

6.1.9 Rank Correlation

Rank method for the computation of the coefficient of correlation is based on the rank or the order & not the magnitude of the variable. Accordingly it is more suitable when the variables can be arranged for e.g. in case of intelligence or beauty or any other qualitative phenomenon. The ranks may range from 1 to *n*. Edward spearman has provided the following formula —

$$R = 1 - \frac{6 \quad D^2}{N(N^2 - 1)}$$

Where N = Number of pairs of variable X & Y

D = Rank difference

Example 4 :

From the data given belows calculate the rank correlation between x & Y

X	78	89	97	69	59	79	68	57
Y	125	137	156	112	107	136	123	108

Solution :

Table : Computation of Rank Correlation

X	Y	R_1	R_2	Rank difference $D = R_1 - R_2$	D^2
78	125	4	4	0	0
89	137	2	2	0	0
97	156	1	1	0	0
69	112	5	6	–1	1
59	107	7	8	–1	1
79	136	3	3	0	0
68	123	6	5	1	1
57	108	8	7	1	1
				$\sum D = 0$	$D^2 = 4$

Rank correlation $= 1 - \frac{6\sum D^2}{N(N^2-1)}$

$$= 1 - \frac{6x4}{8(64-1)} = 1 - \frac{3}{63}$$

$$= 0.95$$

This shows there is very high positive correlation between X & Y.

Example 5 : Calculate Rank Correlation from the following data.

Marks in statistics	10	4	2	5	5	6	9	8
Marks in Maths	10	6	2	5	2	5	9	8

Solution : Table : Calculation of Rank correlation

X	Y	Rank	R_2	D	D^2
10	10	1	1	0	0
4	6	7	4	3	9
2	2	8	7.5	0.5	1
5	5	5.5	5.5	0	0
5	2	5.5	7.5	-2	4
6	5	4	5.5	-1.5	2.25
9	9	2	2	0	0
8	8	3	3	0	0
					$\sum D^2 = 16.25$

$$R = 1 - 6\frac{\left[\sum D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + ...\right]}{N(N^2-1)}$$

Here *m*, m_2 ... denote the number of times ranks are tied in both the variables, the subscripts & denote the first tie, second tie,...., in both the variables

$$= 1 - 6\frac{\left[16.25 + \frac{1}{12}(2^3-2) + \frac{1}{12}(2^3-2)\right]}{8(8^2-1)}$$

$$= 1 - 6\frac{\left[16.25 + \frac{6}{12} + \frac{6}{12}\right]}{8(63)} = 1 - 6\frac{[16.25 + 0.5 + 0.5]}{504}$$

$$= \frac{1 - 6x17.25}{504} = \frac{1 - 103.5}{504}$$

= 1 – 0.205

= 0.795

Example 6 : Find the coefficient of correlation between price and sales from the following data :

Prices (X)	103	98	85	92	90	88	90	94	85
Sales (Y)	500	610	700	630	670	800	570	700	680

Solution : Let the value of assumed mean for X(A_X) be 90

Let the value of assumed mean for Y(A_y) be 700

Table : Calculation of correlation coefficient

X	Y	$d_x = \frac{X - A_x}{1}$ = X–90/1	$d_y = \frac{Y - A_Y}{10}$ = Y – 700/10	d_x^2	d^2y	$d_x d_y$
103	500	13	-20	169	400	-260
98	610	8	-9	64	81	-72
85	700	-5	0	25	0	0
92	630	2	-7	4	49	-14
90	670	0	-3	0	9	0
88	800	-2	10	4	100	-20
90	570	0	-13	0	169	0
94	700	4	0	16	0	0
95	680	5	-2	25	4	-10
		$d_x = 25$	$d_y = -44$	$d_x^2 = 307$	$d_y^2 = 812$	$d_x d_y = -376$

Note : *As r is a pure number, change of scale does not affect its value. Hence the values are divided by 10 in column 4 to make the calculations simple. The following formula can be applied to all the problems.*

$$r = \frac{\sum d_x d_y - \frac{(\sum d_x)(\sum d_y)}{N}}{\sqrt{\sum d_x^2 - \frac{(\sum d_x)^2}{N}}\sqrt{\sum d_y^2 - \frac{(\sum d_y)^2}{N}}} = \frac{-376 - \frac{(25 x - 44)}{10}}{\sqrt{307 - \frac{(25)^2}{10}}\sqrt{812 - \frac{(-44)^2}{10}}}$$

$$= \frac{-376 + 110}{\sqrt{307 - 62.5}\sqrt{812 - 193.6}} = \frac{-266}{\sqrt{244.5}\sqrt{618.4}}$$

$$= \frac{-266}{13.64 \times 24.88}$$

$$= \frac{-266}{389.12}$$

r = 0.684

Example 7 :

Find the coefficient of correlation from the following data and interpret your result

Prices (X)	300	350	400	450	500	550	600	650	700
Sales (Y)	800	900	1000	1100	1200	1300	1400	1500	1600

Solution :

Table : Calculation of correlation coefficient

X	Y	$x = \frac{X-\bar{X}}{50}$	$y = \frac{Y-\bar{Y}}{100}$	x^2	y^2	xy
300	800	-4	-4	16	16	16
350	900	-3	-3	9	9	9
400	1000	-2	-2	4	4	4
450	1100	-1	-1	1	1	1
500	1200	0	0	0	0	0
550	1300	1	1	1	1	1
600	1400	2	2	4	4	4
650	1500	3	3	9	9	9
700	1600	4	4	16	16	16
$\sum X = 4500$	$\sum Y = 10800$	$\sum X = 0$	$\sum y = 0$	$\sum X^2 = 60$	$\sum X^2 = 60$	$xy = 60$

Note that as r is a pure number, change of scale does not affect its value. Hence the values are divided by 50 in column 3 and are divided by 100 in column 4 to make the calculations simple.

$$\bar{X} = \frac{X}{N} = \frac{4500}{9} = 500$$

$$\bar{Y} = \frac{\sum Y}{N} = \frac{10800}{9} = 1200$$

$$r = \frac{xy}{\sqrt{x^2 \cdot \quad y^2}}$$

$$= \frac{60}{\sqrt{60 \times 60}}$$

$$= \frac{60}{60} = 1$$

Since r = +1, there is perfect positive correlation between X and Y.

Example 8 :

The following data gives the distribution of the total population and those who are totally or partially blind among them. Find out Karl Pearson's coefficient of correlation.

Age (in Years)	*No. of persons (in'000)*	*Blind*
15	80	12
16	100	30
17	120	48
18	150	90
19	200	150
20	250	200

Solution :

As we have to find out the correlation between the age of persons and the number of persons who are blinds, we find out percentage of blinds (i.e. blinds per 100 persons of population).

Taking age as X and blinds per 100 persons as Y

Table : Calculation of correlation coefficient

X	*Y*	*x = X – 17.5*	*y = Y – 50*	*xy*	*x²*	*y²*
15	15	-2.5	-35	87.5	6.25	1225
16	30	-1.5	-20	30	2.25	400
17	40	-0.5	-10	5	0.25	100
18	60	0.5	10	5	0.25	100
19	75	1.5	25	37.5	2.25	625
20	80	2.5	30	75	6.25	900
$\sum X=105$	$\sum Y=300$	$X = 0$	$\sum y=0$	$\sum Xy=24$	$\sum X^2=17.5$	$\sum y^2=3350$

$$\overline{X}=\frac{\sum X}{N}=\frac{105}{6}=17.5$$

$$\overline{Y}=\frac{\sum Y}{N}=\frac{300}{6}=50$$

$$r=\frac{\sum xy}{\sqrt{\sum x^2 \sum y^2}}$$

$$r=\frac{240}{\sqrt{17.5x3350}}=0.99$$

There is very high positive correlation between the age of a person & blindne s.

Example 9 :

Calculate the Karl Pearson's coefficient of correlation from the information given below-

- Covariance between two variables X and Y = -15
- Coefficient of variation of X = 25%

- Mean of X =20
- Variance of Y = 16

Solution :

Cov. (X,Y) = -15

$$C.V._x = \frac{\sigma_x}{mean} x 100$$

$$= \frac{\sigma_x}{20} x 100 = 25$$

$$\sigma_x = 5$$

$$(\sigma_y^2) = 16 \text{ (given)}$$

$$\sigma_y = 4$$

$$r = \frac{cov.(X, Y)}{\sigma_x \sigma_y}$$

$$= \frac{-15}{5x4}$$

= – 0.75

Example 10 :

From the following data, compute coefficient of correlation (r) between X and Y:

	X series	Y series
Arithmetic Mean	25	18
Square of Deviations from A.M.	136	138
Summation of products of deviations of X and Y series from their respective means	122	
Number of pairs of values	15	

Solution :

$$r = \frac{\sum xy}{\sqrt{\sum x^2 \sum y^2}}$$

$$r = \frac{122}{\sqrt{136 x 138}}$$

$$r = \frac{122}{136.9}$$

= 0.891

Example 11 :

Calculate Karl Pearson's coefficient of correlation between variables X and Y using the following data:

X	25	40	30	25	10	5	10	15	30	20
Y	10	25	40	15	20	40	28	22	15	5

Solution : Table : Calculation of coefficient of correlation

X	Y	$x = X - 21$	$y = Y - 22$	x^2	y^2	xy
25	10	4	-12	16	144	-48
40	25	19	3	361	9	57
30	40	9	18	81	324	162
25	15	4	-7	16	49	-28
10	20	-11	-2	121	4	22
5	40	-16	18	256	324	-288
10	28	-11	6	121	36	-66
15	22	-6	0	36	0	0
30	15	9	-7	81	49	-63
20	5	-1	-17	1	289	17
$\sum X = 210$	$\sum Y = 220$	$\sum X = 0$	$\sum y = 0$	$\sum x^2 = 1090$	$\sum x^2 = 1228$	$\sum xy = -235$

$$\overline{X} = \frac{\sum X}{N} = \frac{210}{10}$$

$= 21$

$$\overline{Y} = \frac{Y}{N} = \frac{220}{10}$$

$= 22$

$$r = \frac{-235}{\sqrt{1090 \times 1228}}$$

$$= \frac{-235}{1156.94}$$

$= -0.203$

*('Hint - First calculate mean of X series **and** Y series. If they are in integer then **use** Method I. If they are in points then use short cut method ie. Method **II)***

Example 12 :

From the following data, calculate Karl Pearson's coefficient of correlation

Height of fathers (in inches)	66	68	69	72	65	59	62	67	61	71
Height of sons (in inches)	65	64	67	69	64	60	59	68	60	64

Solution : Table Calculation of coefficient of correlation between height of fathers and height of sons

	X	Y	x	y	x^2	y^2	xy
1	66	65	0	1	0	1	0
2	68	64	2	0	4	0	0
3	69	67	3	3	9	9	9
4	72	69	6	5	36	25	30
5	65	64	-1	0	1	0	0
6	59	60	-7	-4	49	16	28
7	62	59	-4	-5	16	25	20
8	67	68	1	4	1	16	4
9	61	60	-5	-4	25	16	20
10	71	64	5	0	25	0	0
	$\sum X=660$	$\sum Y=640$			$\sum x^2=166$	$\sum y^2=108$	$\sum xy=111$

$$\bar{X}=\frac{\sum X}{N}$$

$$=\frac{660}{10}=66\text{inches}$$

$$\bar{Y}=\frac{\sum Y}{N}$$

$$=\frac{640}{10}=64\text{inches}$$

$$r=\frac{\sum Xy}{\sqrt{\sum x^2 \sum y^2}}$$

$$r=\frac{111}{\sqrt{166 \times 108}}$$

$$=0.829$$

Example 13 :

Calculate Karl Pearson's coefficient of correlation from the following information and comment on the result:

Standard deviation of X series	10
Standard deviation of Y series	12
Arithmetic mean of X series	25
Arithmetic mean of Y series	35
Summation of product of deviations from actual arithmetic means of two series	24
Number of observations	20

Solution :

$$r = \frac{\sum xy}{N\sigma_x \sigma_y}$$

where

r = coefficient of correlation

$\sum xy$ = Summation of product of deviations from

actual arithmetic means of two series

σ_x = Standard deviation of X series

σ_y = Standard deviation of Y series

$$r = \frac{24}{20 \times 10 \times 12} = \frac{24}{2400} = 0.01$$

Example 14 :

Calculate the coefficient of correlation by Pearson's method between the density of population and the death rate.

	Area in Sq. Km	*Population in '000*	*No. of deaths*
A	300	60	600
B	360	180	2880
C	200	80	1120
D	120	84	1680
E	240	144	2448
F	160	48	624

Solution :

Using formula

$$\text{Density (X)} = \frac{\text{population}}{\text{area}} \text{ and death rate (Y)} = \frac{\text{No. of dealth}}{\text{population}}$$

Table : Calculation of Density (X) and Death Rate (Y)

	Area in Sq. Km	*Population in '000*	*No. of deaths*	*X*	*Y*
A	300	60	600	200	10
B	360	180	2880	500	16
C	200	80	1120	400	14
D	120	84	1680	700	20
E	240	144	2448	600	17
F	160	48	624	300	13

Table : Calculation of coefficient of correlation

	X	Y	x' = X – 450	x = x'/50	y = Y – 15	x^2	y^2	xy
A	200	10	-250	-5	-5	25	25	25
B	500	16	50	1	1	1	1	1
C	400	14	-50	-1	-1	1	1	1
D	700	20	250	5	5	25	25	25
E	600	17	150	3	2	9	4	6
F	300	13	-150	-3	-2	9	4	6
	$\sum x = 2700$	$\sum Y = 90$		$x = 0$	$\sum y = 0$	$\sum x^2 = 70$	$y^2 = 60$	$\sum xy = 64$

$$\overline{X} = \frac{\sum X}{N}$$

$$= \frac{2700}{6} = 450$$

$$\overline{Y} = \frac{\sum Y}{N} \qquad = \frac{90}{6} = 15$$

$$r = \frac{\sum Xy}{\sqrt{\sum x^2 \sum y^2}}$$

$$r = \frac{64}{\sqrt{70 \text{ x } 60}} \qquad = \frac{64}{\sqrt{4200}} = \frac{64}{64.80}$$

$$= 0.987$$

SELF EXAMINATION QUESTIONS (Theory)

Problem 1. Define correlation explain various types of correlation with suitable example.

Problem 2. State any two of the properties of Pearson's coefficient of correlation.

Problem 3. What is rank correlation explain with the help of an example.

UNSOLVED PROBLEMS (PRACTICAL)

Problem 4. Find correlation coefficient between the variable X and Y

X	80	86	34	56	76	89	65	45	54	15
Y	25	35	40	54	44	25	21	28	20	28

[Ans. r = – 0.047]

Problem 5. Find correlation between age of husband and wife

Age of husband	80	45	55	56	58	60	65	68	70
Age of wife	82	56	50	48	60	62	64	65	70

(Ans. r = 0.862)

Problem 6. The total of the multiplication of deviation of X and Y = 3044
No. of pairs of the observations is 10
Total of deviation of X = (-)170
Total of deviations of Y = (-)20
Total of squares of deviations of X = 8281
Total of squares of deviations of Y = 2264
Find out the coefficient of correlation when the arbitrary means of X and Y are 82 and 68 respectively.

(Ans.: r = 0.781, hint: arbitrary mean are not required for the solution)

Problem 7. Calculate coefficient of correlation from the following data and comment on the result.

Experience X	16	12	18	4	3	10	5	12
Performance Y	23	22	24	17	19	20	18	21

(Ans. 0.951)

Problem 8. Calculate rank correlation coefficient between two seris X and Y, given below –

X	70	65	71	62	58	69	78	64
Y	91	76	65	83	90	64	55	48

(Ans. -0.309)

Problem 9. Calculate rank correlation coefficient from the data given below –

X	75	88	95	70	60	80	81	50
Y	120	134	150	115	110	140	142	100

(Ans. 0.929)

IMPORTANT FORMULAE

KARL PEARSON'S COEFFICIENT OF CORRELATION

A. WHEN DERIVATIONS ARE TAKEN FROM ACTUAL MEAN

$$r = \frac{\sum xy}{N\sigma_x\sigma_y}$$

$$r = \frac{\sum xy}{\sqrt{\sum x^2 \, x \sum y^2}}$$

B. WHEN DEVIATIONS ARE TAKEN FROM ASSUMED MEAN

$$r = \frac{\sum d_x d_y - \frac{(\sum d_x)(\sum d_y)}{N}}{\sqrt{\sum d_x^2 - \frac{(\sum d_x)^2}{N}}\sqrt{\sum d_y^2 - \frac{(\sum d_y)^2}{N}}}$$

RANK CORRELATION

$$R = 1 - \frac{6\sum D^2}{N(N^2-1)}$$

$$R = 1 - 6 = \frac{\left[\sum D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \ldots\right]}{N(N^2-1)}$$

6.2 REGRESSION ANALYSIS

6.2.1. INTRODUCTION

In the last chapter we studied the concept of statistical relationship between two variables such as - amount of fertilizer used and yield of a crop; price of a product and its supply, level of sales and amount of advertisement and so on.

The relationship between such variables do indicate the degree and direction of their association, but they do not answer the question that whether there is any functional (or algebraic) relationship between two variables? If yes, can it be used to estimate the most likely value of one variable, given the value of other variable?

> "In regression analysis we shall develop an estimating equation i.e., a mathematical formula that relates the known variables to the unknown variable. (Then, after we have learned the pattern of this relationship, we can apply correlation analysis to determine the degree to which the variables are related. Correlation analysis, then, tells us how well the estimating equation actually describes the relationship). The variable which is used to predict the unknown variables is called the 'independent' or 'explaining' variable, and the variable whose value is to be predicted is called the 'dependent' or 'explained' variable." Ya-lun Chou

6.2.2. DISTINCTION BETWEEN CORRELATION AND REGRESSION

By correlation we mean the degree of association or relationship between two or more variables. Correlation does not predict anything about the cause & effect relationship. Even a high degree of correlation does not imply necessarily that a cause & effect relationship exists between the two variables.

Whereas in case of regression analysis, there is a functional relationship between Y and X such that for each value of Y there is only one value of X. One of the variables is identified as a dependent variable the other(s) as independent valuable(s). The expression is derived for the purpose of predicting values of a dependent variable on the basis of independent valuable(s).

6.2.3. REGRESSION LINES

A regression line is the line which shows the best mean values of one variable correspond-ing to mean values of the other. With two series X and Y, there are two arithmetic regression lines, one showing the best mean values of X corresponding to mean Y's and the other showing the best mean values of Y corresponding to mean X's. In the context of scatter diagram, the regression line is the straight line that best fits the scatter diagram. The most commonly used criteria is that it is the straight line that minimise the sum of the squared deviations between the predicted and observed values of the dependent variable. In the case of two variables X and Y, there will be two regression lines as the regression of X on Y and regression of Y on X.

6.2.4. REGRESSION EQUATIONS

There are different methods of deriving regression equations

(1) By taking actual values of X and Y

(2) By taking deviations from actual mean

(3) By taking deviations from assume mean

6.2.5. METHOD I WHEN ACTUAL VALUES ARE TAKEN

The regression equation of Y on X is expressed as follows:

$Y_c = a+bX$

Where *a* and *b* can be found out by solving the following two normal equations simultaneously:

$$\sum Y = Na + b\sum X$$

$$\sum XY = a\sum X + b\sum X^2$$

The regression equation of X on Y is expressed as follows:

$X_c = a + bY$

Where *a* and *b* can be found out by solving the following two normal equations simultaneously:

$$\sum X = Na + b\sum Y$$

$$\sum XY = a\sum Y + b\sum Y^2$$

Example 15 : From the following table find :

(1) Regression Equation of X on Y.

(2) Regression Equation of Y on X.

X	10	12	18	22	25	9
Y	15	18	21	26	32	8

Solution : Table : Calculation of Regression Equations

X	y	x^2	Y^2	XY
10	15	100	225	150
12	18	144	324	216
18	21	324	441	378
22	26	484	676	572
25	32	625	1024	800
9	8	81	64	72
$\sum X = 96$	$\sum Y = 120$	$\sum X^2 = 1758$	$\sum Y^2 = 2754$	$\sum XY = 2188$

Regression equation of X on Y is given by :

$X = a + bY$

Where a & b can be found out by solving the following 2 equations simultaneously –

$\sum X = Na + b \sum Y$

$\sum XY = a\sum Y + b\sum Y^2$

Substituting the values obtained from the table above, we get

$96 = 6a + 120\,b$ ....(1)

$2188 = 120a + 2754b$ ....(2)

Multiply equation 1 by 20 & subtract equation 2 from it.

$$
\begin{array}{rcl}
1920 & = & 120a + 2400\,b \\
\underset{-}{2188} & = & \underset{-}{}120a \underset{-}{} 2754\,b \\
\hline
-268 & = & 0 \quad -354\,b
\end{array}
$$

$$b = \frac{-268}{-354}$$

$b = 0.76$

Put this value of b in eq .... (1)

$96 = 6a + 120 \times 0.76$

$96 = 6a + 91.2$

$6a = 96 - 91.2$

$$a = \frac{4.8}{6} = 0.8$$

Put the value a & b in the regression equation of X on Y

$X = a + by$

$X = 0.8 + 0.76Y$

Regression equation of Yon X is given by

$Y = a + bX$

Where constants *a* and *b* can be found out by solving the following 2 normal equations simultaneously—

$\sum Y = Na + b\sum X$

$\sum XY = a\sum X + b\sum X^2$

Substituting the value obtained from the above table, we get

$120 = 6a + 96b$ ....(1)

$2188 = 96a + 1758b$ ....(2)

Multiply e.g. 1 by 16 & subtract equation 2 from it

$1920 = 96a + 1536$

$2188 = 96a + 1758$

$- \quad - \quad -$

$-268 = 0 + -222b$

$$b = \frac{268}{222} = 1.21$$

Put the value of b in equation 1

$120 = 6a + 96 \times 1.21$

$120 = 6a + 116.16$

$6a = 120 - 116.16$

$6a = 3.84$

$$a = \frac{3.84}{6} = 0.64$$

Put the value of *a* and 6 in the regression equation of Y on X

$Y = a + bX$

$Y = 0.64 + 1.21X$

There is an alternative method of finding the regression equations. Instead of the normal equations, deviations from the respective arithmetic mean or assumed mean are considered :

6.2.6. METHOD II WHEN DEVIATIONS ARE TAKEN FROM ACTUAL MEAN

Regression equation of X and Y is given by

$$X - \bar{X} = b_{XY}(Y - \bar{Y})$$

where $\bar{X}$, $\bar{Y}$ are actual mean of X & Y series respectively

$$b_{XY} = \frac{XY}{Y^2}$$

$\sum XY$ = Sum of product of deviations taken from actual mean of X & Y.

$\sum Y^2$ = Sum of sequare of deviations from actual mean of Y.

Regression equation of Y and X is given by

$Y - \bar{Y} = b_{yx}(X - \bar{X})$

Where $b_{YX} = \frac{XY}{X^2}$

ΣXY = Sum of product of deviations taken from actual mean of X & Y

ΣX^2 = Sum of square of deviations from actual mean of X.

Example 16 : From the data of the previous example find the two regression equation by taking deviations from actual mean.

Soluation :

Table : Calculation Regression Equations

X	Y	$X-\bar{X}=x$	$Y-\bar{Y}=Y$	x^2	Y^2	XY
10	15	-6	-5	36	25	30
12	18	-4	-2	16	4	8
18	21	2	1	4	1	2
22	26	6	6	36	36	36
25	32	9	12	81	144	108
9	8	-7	-12	49	144	84
$\Sigma X = 96$	$\Sigma Y = 120$	0	0	$\Sigma X^2 = 222$	$\Sigma Y^2 = 354$	$\Sigma XY = 268$

$\bar{X} = \frac{96}{6} = 16$

$\bar{Y} = \frac{120}{6} = 20$

Regression equation of x on Y

$X - \bar{X} = b_{XY}(Y - \bar{Y})$

Where $b_{XY} = \frac{XY}{Y^2}$

$b_{XY} = \frac{268}{354} = 0.76$

putting the value of b_{YX} in the above equation & also put $\bar{X} = 16 \& \bar{Y} = 20$

$X - 16 \quad = 0.76\,(Y - 20)$

$X - 16 \quad = 0.76Y - 15.2$

$X \quad = 0.76Y - 15.2 + 16$

$X \quad = 0.76Y + 0.8$

Regression equation of Y on X

$$Y - \overline{Y} = b_{YX}(x - \overline{x})$$

Where $b_{YX} = \frac{XY}{x^2}$

$$b_{YX} = \frac{268}{222} = 1.21$$

Putting the value of b_{YX} in above equation & also put $\overline{Y} = 20$ & $\overline{X} = 16$

Y - 20 = 1.21(X - 16)

Y - 20 = 1.21X - 1.21X16

Y - 20 = 1.21X -19.36

Y = 1.21X -19.36+ 20

Y = 1.21X +0.64

6.2.7. METHOD III WHEN DEVIATIONS ARE TAKEN FROM ASSUMED MEAN

In case the actual mean of the respective series are not an integer but are in decimals, it becomes cumbersome to calculate deviations from actual mean as all the values so calculated would also be in points. In such a case deviations are taken from assumed mean to simplify the calculations.

Regression equation of X on Y is given by

$$X - \overline{X} = b_{XY}(Y - \overline{Y})$$

$$\text{Where } b_{XY} = \frac{d_X d_Y - \quad d_X \quad d_Y}{d_Y^2 \quad \frac{(\quad d_Y)^2}{N}}$$

Regression equation of Y on X is given as

$$Y - \overline{Y} = b_{YX}(X - \overline{X})$$

$$\text{Where } b_{YX} = \frac{d_X d_Y - \quad d_X \quad d_Y}{d_X^2 \quad \frac{(\quad d_X)^2}{N}}$$

Example 17 :

Calculate 2 regression equations from the following table—

X	10	12	15	19	15
Y	12	15	25	35	14

Solution : Table : Calculation Regression Equations

X	Y	d_x	d_y	d_x^2	d_y^2	d_xd_y
10	12	-5	-13	25	169	65
12	15	-3	-10	9	100	30
15	25	0	0	0	0	0
19	35	4	10	16	100	40
15	14	0	-11	0	121	0
$\Sigma X = 71$	$\Sigma Y = 101$	$\Sigma d_X = -4$	$\Sigma d_Y = -24$	$\Sigma d_X^2 = 50$	$\Sigma d_Y^2 = 490$	$\Sigma d_Xd_Y = 135$

$$\overline{X} = \frac{71}{5} = 14.2$$

$$\overline{Y} = \frac{101}{5} = 20.2$$

Since $\overline{X}$ & $\overline{Y}$ are not an integer we would solve it by taking assume mean of 15 from X series, and 25 from Y series

REGRESSION EQUATION OF X ON Y

$$X - \overline{X} = b_{XY}(Y - \overline{Y})$$

$$b_{XY} = \frac{d_Xd_Y - \quad d_X \quad d_Y}{d_Y^2 \quad \frac{\left(\quad d_Y\right)^2}{N}}$$

By putting the values from the above table we get

$$b_{XY} = \frac{135 - (-4) \times (-24)}{490 - \frac{(-24)^2}{5}}$$

$$= \frac{135 - 96}{490 - \frac{576}{5}}$$

$$= \frac{39}{490 - 115.2} = \frac{39}{374.8} = 0.104$$

$$X - \overline{X} = b_{XY}(Y - \overline{Y})$$

$X - 14.2 = 0.104\ (Y - 20.2)$

$X - 14.2 = 0.104Y - 2.10$

$X = 0.104Y - 2.10 + 14.2$

$X = 0.104Y + 12.1$

Regression equation of Yand X

$$Y-\overline{Y}=b_{YX}(x-\overline{x})$$

$$\text{Where } b_{YX}=\frac{d_Xd_Y - \quad d_X \quad d_Y}{d_X^2 \quad \frac{\left(\quad d_X\right)^2}{N}}$$

$$b_{YX}=\frac{135-(-4)(-24)}{50-\frac{(-4)^2}{5}}$$

$$=\frac{135-96}{50-\frac{16}{5}}$$

$$=\frac{39}{50-3.2}=\frac{39}{46.8}=0.83$$

$$Y-\overline{Y}=b_{YX}(X-\overline{X})$$

$Y - 20.2 \quad = 0.83\ (X - 14.2)$

$Y - 20.2 \quad = 0.83\ X - 0.83\ X\ 14.2$

$Y = 0.83X - 11.78 + 20.2$

$Y = 0.83X + 8.42$

6.2.8. REGRESSION COEFFICIENTS

The regression coefficient gives the value by which one variable increases for a unit increase in other variable, b_{XY} and b_{YX} are two coefficient of regression.

***Regression coefficient of x on y* or b_{xy}** can be calculated by any of the following ways—

(1) when standard deviation are given

$$b_{XY} = r\frac{\sigma X}{\sigma Y}$$

(2) when deviations are taken from actual mean

$$b_{XY} = \frac{\quad XY}{\quad Y^2}$$

(3) when deviations are taken from assumed mean

$$b_{XY}=\frac{d_Xd_Y - \quad d_X \quad d_Y}{d_Y^2 \quad \frac{\left(\quad d_Y\right)^2}{N}}$$

Regression coefficient of y on x or b_{yx} can be calculated by any of the following ways (Note that X & Y has been interchanged in the above formulas)—

(1) when standard deviations are given-

$$b_{YX} = r\frac{\sigma Y}{\sigma X}$$

(2) when deviations are taken from actual mean

$$b_{YX} = \frac{XY}{X^2}$$

(3) when deviations are taken from assumed mean

$$b_{YX} = \frac{d_X d_Y - \quad d_X \quad d_Y}{d_X^2 - \frac{(\quad d_X)^2}{N}}$$

6.4.8.1. FEATURES OF REGRESSION COEFFICIENTS

(i) Both of regression coefficients should have same sign i e., either positive or negative.

(2) Coefficient of correlation could be found out if regression coefficients are known; by the formula

$$r = \sqrt{b_{XY} x\, b_{YX}}$$

(3) Correlation coefficient would have the same sign as that of regression coefficients. ie., either positive or negative.

(4) Since $-1 \le r \le 1$ this implies both the regression coefficient cannot be greater than one.

For example if b_{XY} = 2 and b_{YX} = 1.5 then value of $r = \sqrt{2 x 1.5} = \sqrt{3} = 1.732$, which is not possible.

Example 18 : Given:

$N = 5 \quad \bar{X} = 20 \quad \bar{Y} = 10$

$\Sigma(X - 20)^2 = 100 \qquad \Sigma(Y - 10)^2 = 60$

$\Sigma(X-20)(Y-10) = 40$

Find two regression equations.

Solution :

$N = 5$

$\bar{X} = 20$

$\bar{Y} = 10$

$\Sigma(X - 20)^2 = \Sigma(X - \bar{X})^2 = \Sigma X^2 = 100$

$\Sigma(Y - 10)^2 = \Sigma(Y - \bar{Y})^2 = \Sigma Y^2 = 60$

$\sum(X-20)(Y-10) = \sum xy = 40$

Regression Equation of X and Y

$X - \bar{X} = b_{XY}(Y - \bar{Y})$

Where $b_{XY} = \frac{XY}{Y^2} = \frac{40}{60} = 0.667$

$X - 20 = 0.667(Y-10)$

$X - 20 = 0.667Y - 6.67$

$X = 20 + 0.667Y - 6.67$

$X = 0.667Y + 13.33$

Regression Equation of Y and X

$Y - \bar{Y} = b_{YX}(X - \bar{X})$

Where $b_{YX} = \frac{XY}{X^2} = \frac{40}{100} = 0.4$

$Y - 10 = 0.4(X - 20)$

$Y - 10 = 0.4X - 8$

$Y = 10 + 0.4X - 8$

$Y = 0.4X + 2$

$r = \sqrt{b_{XY} b_{YX}}$

$= \sqrt{0.4 \times 0.667}$

$= \sqrt{0.2668}$

$= 0.517$

Example 19 :

Following are the marks in Maths and English

	Maths	*English*
Mean	40	50
Standard Deviation	10	16
Coefficient of correlation	0.5	

(1) Find two regression equation

(2) Find the most likely marks in Maths if marks in English are 40.

Solution : Let the marks in Maths be denoted by X and the marks in English by Y.

We have: $\bar{X} = 40$

$\bar{Y} = 50$

$\sigma_X = 10$

$\sigma_Y = 16$

$r = 0.5$

Regression Equation of Y on X

$$Y - \bar{Y} = r\frac{\sigma_Y}{\sigma_X}(X - \bar{X})$$

Regression Equation of X on Y

$$X - \bar{X} = r\frac{\sigma_X}{\sigma_Y}(Y - \bar{Y})$$

Regression Equation of Y on X

$$Y - \bar{Y} = r\frac{\sigma_Y}{\sigma_X}(X - \bar{X})$$

$$Y - 50 = 0.5\,\frac{16}{10}\,(X - 40)$$

$Y - 50 = 0.8\,(X - 40)$

$Y - 50 = 0.8X - 32$

$Y - 50 = 0.8X - 32$

$Y = 18 + 0.8X$

Regression Equation of X on Y

$$X - \bar{X} = r\frac{\sigma_X}{\sigma_Y}(Y - \bar{Y})$$

$$X - 40 = 0.5\,\frac{10}{16}\,(Y - 50)$$

$X - 40 = 0.3125\,(Y - 50)$

$X = 40 + 0.3125Y - 15.625$

$X = 24.375 + 0.3125Y$

To find likely marks in Maths if marks in English are 40, put Y = 40 in regression equation of X on Y.

$X = 0.3125\,(40) + 24.375$

$= 12.5 + 24.375$

$= 36.875$

Example 20 :

The data about the sales and advertisement expenditure of firm are given below:

	Sales (₹ in crores)	Advertisement Expenditure (₹ in crores)
Mean	40	6
Standard deviation	10	1.5
Coefficient of Correlation	0.9	

(i) Estimate the likely sales for a proposed advertisement expenditure of ₹ 10 crores.

(ii) What should be the advertisement expenditure if the firm proposes a sales target of ₹ 60 crores?

Solution :

Let the sales be denoted by X and advertisement expenditure by Y.

We have $\overline{X} = 40$

$\overline{Y} = 6$

$\sigma_X = 10$

$\sigma_Y = 1.5$

$r = 0.9$

Regression Equation of Y on X

$$Y - \overline{Y} = r\frac{\sigma_Y}{\sigma_X}(X - \overline{X})$$

(i) To estimate the likely sales for a proposed advertisement expenditure of ₹ 10 crores, we have to find regression equation of X on Y.

Regression Equation of X on Y

$$X - \overline{X} = r\frac{\sigma_X}{\sigma_Y}(Y - \overline{Y})$$

$$X - 40 = 0.9\frac{10}{1.5}(Y - 6)$$

$X - 40 = 6(Y - 6)$

$X - 40 = 6Y - 36$

$X = 40 + 6Y - 36$

$X = 6Y + 4$

Putting Y = 10 in above equation

$X = 6 \times 10 + 4 = 64$

Hence, estimated sales = ₹ 64 crores.

(ii) To estimate the advertisement expenditure if the firm proposed as sales target of ₹ 60 crores, we find regression equation of Y on X.

Regression Equation of Y on X :

$$Y - \overline{Y} = r\frac{\sigma_Y}{\sigma_X}(X - \overline{X})$$

$$Y - 6 = 0.9\ \frac{1.5}{10}\ (X - 40)$$

Y - 6 = 0.135 (X-40)

Y - 6 = 0.135 X -5.4

Y = 6 + 0.135X - 5.4

Y = 0.6 + 0.135X

Put X = 60 in regression equation of Y on X.

Y = 0.6 + 0.135 (60)

Y = 0.6 + 8.10

Y = 8.7 crore

Example 21 : Given:

Covariance between X and Y = 16

Variance of X = 25

Variance of Y =16

(i) Calculate coefficient of correlation between X and Y

(ii) If arithmetic means of X and Y are 20 and 30 respectively, find regression equation of Y on X.

(iii) Estimate Y when X = 30

Solution :

Given covariance between X and Y = $\frac{\sum XY}{N}$ = 16

Variance of X = σ_x^2 = 25

$\sigma_x = \sqrt{25}$ = 5

Variance of Y = σ_Y^2 = 16

$\sigma_Y = \sqrt{16}$ = 4

Applying formula $r = \frac{XY}{N} \cdot \frac{16}{\sigma_X \times \sigma_Y}$

$$= \frac{16}{5 \times 4} = 0.8$$

(ii) given

$\overline{X}$ = 20

$\overline{Y}$ = 30

Regression Equation of Y on X

$$Y - \overline{Y} = r\frac{\sigma_Y}{\sigma_X}(X - \overline{X})$$

$$Y - 30 = 0.8\ \frac{4}{5}\ (X - 20)$$

Y - 30 = 0.64 (X - 20)

Y - 30 = 0.64X - 12.8

Y = 0.64X +17.2

(iii) Estimate Y when X = 30

Put X = 30 in regression equation of Y on X.

Y = 0.64(30) + 17.2

Y = 17.2 + 19.2

Y = 36.4

Example 22 :

The line of regression of marks in statistics (X) on marks in accountancy (Y) for a class of 50 students is 3Y - 5X + 180 = 0. Average marks in accountancy is 44 and variance of marks in statistics is $\frac{9}{16}^{th}$ of variance of marks in accountancy. Find:

(i) Average marks in statistics

(ii) Coefficient of correlation between X and Y

Solution :

(i) Regression equation X on Y is

3Y - 5X +180 = 0

5x = 3Y + 180

$$X = \frac{3}{5}Y + \frac{180}{5} = 0.6Y + 36$$

$$b_{XY} = \frac{3}{5} = 0.6$$

given $\overline{Y}$ = 44, X can be obtained by putting Y = 44 in Regression Equation of X on Y

X = 0.6Y + 36

X = 0.6(44) + 36

X = 26.4 + 36

X = 62.4

X = 62.4 marks

Average marks in statistics are 62.4

(ii) given variance of marks in statistics (X) is $\frac{9}{16}$ th of variance of marks in accountancy (Y)

i.e., $\sigma_X^2 = \frac{9}{16}\sigma_Y^2$

$\frac{\sigma_X^2}{\sigma_Y^2} \quad \frac{9}{16}$

$\frac{\sigma_X}{\sigma_Y} \quad \sqrt{\frac{9}{16}}$

$\frac{\sigma_X}{\sigma_Y} = \frac{3}{4}$

$b_{XY} = r\frac{\sigma_X}{\sigma_Y}$

$0.6 = r\left(\frac{3}{4}\right)$

$r = \frac{0.6x4}{3}$

r = 0.8

Example 23 :

Following data relates to marks in accounts and statistics in B. Com. (Hons.) I Year Examination of a particular year in University of Delhi.

	Accounts	*Statistics*
Mean	30	35
Standard deviation	10	7
Coefficient of correlation	0.8	

Find two regression equations and calculate the expected marks in accounts if marks secured by a student in statistics are 40

Solution :

Let the marks in accounts be denoted by X and the marks in statistics by Y.

we have

$\bar{X} = 30$

$\bar{Y} = 35$

$\sigma_X = 10$

$\sigma_Y = 7$

$r = 0.8$

Regression Equation of Y on X

$$Y - \overline{Y} = r\frac{\sigma_Y}{\sigma_X}(X - \overline{X})$$

$$Y - 35 = 0.8\ \frac{7}{10}\ (X - 30)$$

Y - 35 = 0.56 (X- 30)

Y - 35 = 0.56X - 16.8

Y = 0.56X - 16.8 + 35

Y = 0.56X + 18.2

Regression Equation of X on Y

$$X - \overline{X} = r\frac{\sigma_X}{\sigma_Y}(Y - \overline{Y})$$

$$X - 30 = 0.8\frac{10}{7}(Y - 35)$$

X - 30 = 1.14 (Y - 35)

X - 30 = 1.14Y - 39.9

X = 1.14Y - 39.9 + 30

X = 1.14Y - 9.9

To find likely marks in accounts if marks in statistics are 40, put Y = 40 in regression equation of X on Y.

X = 1.14 (40) -9.9

X = 45.6 - 9.9

= 35.7

Marks in accounts = 35.7

Example 24 :

By using the following data, find out the two lines of regression and from them compute the Karl Pearson's coefficient of correlation.

$\sum X = 250, \sum Y = 300, \sum XY = 7900, \sum X^2 = 6500, \sum Y^2 = 10000, N = 10$

Solution : Regression line of X on Y is :

$X - \overline{X} = b_{XY}(Y - \overline{Y})$

Where,

$$b_{XY} = \frac{N\ \ XY - \ \ X\ \ Y}{N\ \ Y^2 - (\ \ Y)^2},\ \overline{X} = \frac{\ \ X}{N}\ \text{and}\ \overline{Y} = \frac{\ \ Y}{N}$$

$\overline{X} = \frac{250}{10} = 25$ and $\overline{Y} = \frac{300}{10} = 30$

$$b_{XY} = \frac{10(7900) - (250)(300)}{10(10000) - (300)^2} = 0.4$$

∴ Regression line of X on Y is

X - 25 = 0.4(Y - 30)

X = 0.4 Y - 12 + 25

X = 0.4Y + 13

∴ Regression line of Y on X is

$Y - \overline{Y} = b_{XY}(X - \overline{X})$

$$b_{YX} = \frac{N \quad XY - \quad X \quad Y}{N \quad X^2 - (\quad X)^2}$$

$$= \frac{10(7900) - (250)(300)}{10(6500) - (250)^2} = 1.6$$

∴ Regression line of Y on X is

Y - 30 = 1.6 (X - 25)

Y = 106 X - 40 + 30

Y = 106 X - 10

Now $r = \sqrt{b_{XY} \times b_{YX}}$

$= \sqrt{0.4 \times 1.6}$

= 0.8

(Since both b_{YX} and b_{XY} are positive)

Example 25 :

Calculate

(i) Two regression coefficients

(ii) Coefficient of correlation

(iii) Two regression equation from the following information:

$N = 10$ $\quad \sum X = 350$ $\quad \sum Y = 310$

$\sum(X - 35)^2 = 162$ $\quad \sum(Y - 31)^2 = 222$

$\sum(X - 35)(Y - 31) = 92$

Solution :

(1) $\sum X = 350$ N = 10

$$\overline{X} = \frac{X}{N} = \frac{350}{10} = 35$$

$$\overline{Y} = \frac{Y}{N} = \frac{310}{10} = 31$$

Therefore X - 35 and Y - 31 and deviations taken from actual mean

X - 35 = X

Y - 31= Y

Then $\sum X^2 = 162$, $\sum Y^2 = 222$

$\sum XY = 92$

Two regression coefficients are

$$b_{XY} = \frac{XY}{Y^2} = \frac{92}{222} = 0.41$$

$$b_{YX} = \frac{XY}{X^2} = \frac{92}{162} = 0.567 = 0.57$$

(2) $r = \sqrt{b_{XY} \times b_{YX}}$

$= \sqrt{0.41 \times 0.57}$

= 0.48

$\therefore$ Coefficient of correlation is 0.48

(3) Regression equation of X on Y

$X - \overline{X} = b_{XY}(Y - \overline{Y})$

X - 35 = 0.41 (Y-31)

X = 0.41Y -12.71+ 35

X = 0.41Y +22.29

Regression equation of Y on X.

$Y - \overline{Y} = b_{YX}(X - \overline{X})$

Y - 31 = 0.57 (X - 35)

Y = 0.57X - 19.95 + 31

Y = 0.57X + 11.05

Example 26 :

You are given the following information regarding a distribution—

N = 5, $\overline{X} = 10$, $\overline{Y} = 20$, $\sum(X - 4)^2 = 100$

$\sum(Y - 10)^2 = 160$, $\sum(X - 4)(Y - 10) = 80$

Find two regression coefficients

Solution :

Given

A_X= Assume mean of X series = 4

A_Y = Assume mean of Y series =10

N = 5

$\bar{X}$ =10

$\bar{Y}$ =20

Σd_X = sum of deviations taken from assumed mean of x series = Σ(X - 4) = ?

Σd_Y = sum of deviations taken from assumed mean of y series = Σ(Y - 10) = ?

$\Sigma d_X^2 = 100, \; \Sigma d_Y^2 = 160 \;\; \Sigma d_X d_Y = 80$

we know $\bar{X} = A_X + \frac{d_X}{N}$

$$10 = 4 + \frac{d_X}{5}$$

$$10 - 4 = \frac{d_X}{5}$$

$$d_X = 6 \times 5$$

$$d_X = 30$$

$$\bar{Y} = A_Y + \frac{d_Y}{N}$$

$$20 = 10 + \frac{d_Y}{5}$$

$$20 - 10 = \frac{d_Y}{5}$$

$$5 \times 10 = d_Y$$

$$d_Y = 50$$

$$b_{XY} = \frac{d_X d_Y - \frac{d_X \;\; d_Y}{N}}{d_Y^2 - \frac{(\;\; d_Y)^2}{N}}$$

$$= \frac{80 - \frac{30 \times 50}{5}}{160 - \frac{(50)^2}{5}}$$

$$= \frac{80 - \frac{1500}{5}}{160 - \frac{2500}{5}} = \frac{80 - 300}{160 - 500}$$

$$= \frac{-220}{-340} = 0.647$$

$$b_{YX} = \frac{d_X d_Y - \frac{d_X \quad d_Y}{N}}{d_X^2 - \frac{\left(\quad d_X \right)^2}{N}}$$

$$= \frac{80 - \frac{30x50}{5}}{100 - \frac{(30)^2}{5}}$$

$$= \frac{-220}{100 - \frac{900}{5}} = \frac{-220}{100 - 180} = \frac{-220}{-80} = 2.75$$

Example 27 :

Given that ∑X = 120, ∑Y = 432, ∑XY = 4992, ∑X^2 = 1392, ∑Y^2 = 18252 N = 12 Find:

(1) The two regression equations

(2) The regression coefficients

(3) Coefficient of correlation

Solution :

$$\bar{X} = \frac{\sum X}{N} = \frac{120}{12} = 10$$

$$\bar{Y} = \frac{\sum Y}{N} = \frac{432}{12} = 36$$

$$b_{XY} = \frac{XY - \frac{X \quad Y}{N}}{Y^2 - \frac{\left(\quad Y \right)^2}{N}}$$

$$= \frac{4992 - \frac{120 \times 432}{12}}{18252 - \frac{(432)^2}{12}}$$

$$= \frac{4992 - \frac{51840}{12}}{18252 - \frac{186624}{12}}$$

$$= \frac{4992 - 4320}{18252 - 15552} = \frac{672}{2700} = 0.249$$

$$b_{YX} = \frac{XY - \frac{X \quad Y}{N}}{X^2 - \frac{(\quad X)^2}{N}}$$

$$= \frac{4992 - \frac{120 \times 432}{12}}{1392 - \frac{(120)^2}{12}}$$

$$= \frac{672}{1392 - \frac{14400}{12}} = \frac{672}{1392 - 1200} = \frac{672}{192} = 3.5$$

Therefore, the two regresion coefficients are :

$b_{XY} = 0.249$, $b_{YX} = 3.5$

$$r = \sqrt{b_{XY} \times b_{YX}} = \sqrt{0.249 \times 3.5}$$

$$= \sqrt{0.8715} = 0.933$$

Regression equation of X on Y is

$$X - \bar{X} = b_{YX}(Y - \bar{Y})$$

X - 10 = 0.249 (Y - 36)

X = 0.249Y - 8.964 + 10

X = 0.249Y + 1.036

Regression equation of Y on X is

$$(Y - \bar{Y}) = b_{XY}(X - \bar{X})$$

Y - 36 = 3.5 (X - 10)

Y = 3.5X - 35 + 36

Y = 3.5X + 1

SELF EXAMINATION QUESTION

Problem 1. For a given set of data, the following result are available-

$\overline{X} = 53, \overline{Y} = 28, b_{XY} = -1.5 b_{YX} = -0.2$

Find

(1) The two regression equations
(2) The coefficient of correlation
(3) The most likely value of Y when X = 60
(4) The most likely value of X when Y = 20

[**Ans.** : Reg. equation of X on Y : X = 58.6 - 0.2Y, equation of Y on X : Y = 107.5 1.5 - X, r = -0.548; 17.5; 54.6]

Problem 2. From the following data find:

(a) two regression equations
(b) the coefficient of correlation between the marks in economics and statistics
(c) the most likely marks in stastistics when marks in economics are 30

Marks in Economics	25	28	35	32	31	36	29	38	34	32
Marks in statistic	43	46	49	41	36	32	31	30	33	39

[**Ans.** : X = 0.23Y + 40.74, Y = 0.66X + 59.12; r = -0.389, marks in stats: 39.3]

Problem 3. Using the following data obtain two regression equations:

X	16	21	26	23	28	24	17	22	21
Y	33	38	50	39	52	47	35	43	41

[**Ans.** : X = -1.39+0.557Y, Y = 6.624+1.608X]

Problem 4. The following data about the sale and advertisement expenditure of a firm are given below:

	Sales (₹ crores)	Expenditure (₹ crores)
Mean	40	6
S.D.	10	1.5

Coefficient of correlation r = 0.9

(1) Find the likely sales for a proposed expenditure of ₹ 10 crores.
(2) What should be the advertisement expenditure if the firm proposes a sales target of ₹ 60 crores.

[**Ans.:** ₹ 64 crores, ₹ 8.7 crores]

Problem 5. Find the mean of X and Y variables and the coefficient of correlation between them from the following regression equations:

2Y - X = 50, 3Y - 2X = 10

[**Ans.** 130, 90. r = 0.86]

Problem 6. The equations of regression lines between two variables are expressed as 2X - 3Y = 0 and 4Y - 5X - 8 = 0. Find $\overline{X}$ and $\overline{Y}$, the regression coefficients and the correlation coefficient between X and Y.

[**Ans.** (i) $\overline{X}$ = -3.42, $\overline{Y}$ = -2.28, b_{YX} = 0.67, b_{XY} = 0.8, r = 0.73]

Problem 7. Given the mean of X and Y are 35 and 67. Their standard deviations are 2.5 and 3.5 respectively and the coefficient of correlation between them is 0.8.

(i) write down the two regression lines

(ii) obtain the best estimate of X, when Y = 70.

[**Ans.** (i) X = 26.81 + 0.57Y, Y = -5.8 + 1.12X, 66.71]

IMPORTANT FORMULA

REGRESSION EQUATIONS

A. WHEN ACTUAL VALUES ARE TAKEN

The regresssion equation of Y on X is expressed as follows:

$Y_c = a + bX$

$\sum y = Na + b\sum X$

$\sum XY = a\sum X + b\sum X^2$

The regression equation of X on Y is expressed as follows:

$X_c = a + bY$

$\sum X = Na + b\sum Y$

$\sum XY = a\sum Y + b\sum Y^2$

B. WHEN DEVIATIONS ARE TAKEN FROM ACTUAL MEAN

Regresssion equation of an X is given by

$$X - \bar{X} = b_{XY}(Y - \bar{Y})$$

$$b_{XY} = \frac{xy}{y^2}$$

Regression equation of an X is given by

$$Y - \bar{Y} = b_{YX}(X - \bar{X})$$

Where $b_{YX} = \frac{xy}{x^2}$

C. WHEN DEVIATIONS ARE TAKEN FROM ASSUMED MEAN

Regression equation of X on Y is given by

$$X - \bar{X} = b_{XY}(Y - \bar{Y})$$

Where $b_{XY} = \dfrac{d_X d_Y - \quad d_X \quad d_Y}{d_Y^2 - \dfrac{(\quad d_Y)^2}{N}}$

Regression equation of Y on X is given by

$$Y - \bar{Y} = b_{YX}(X - \bar{X})$$

Where $b_{YX} = \dfrac{d_X d_Y - \quad d_X \quad d_Y}{d_X^2 - \dfrac{(\quad d_X)^2}{N}}$

Study Note - 7

INDEX NUMBERS

This Study Note includes

INTRODUCTION

An index number is a 'relative number' which expresses the relationship between two variables or two groups of variables where one of the group is used as base

"An index number is a statistical measure designed to show changes invariable or a group of related variables with respect to time, geographic location or other characteristics." - Spiegel

"Index Numbers are devices for measuring difference in the magnitude of a group of related variables" - Croxton and Cowden

"An index number is a statistical measure of fluctuation in a variable arranged in the form of a series and a base period for making comparisons" - L.J. Kaplass

Index number is a statistical device designed to measure changes or differences in magnitudes in a variable or group of related variables with respect to time, geographic location or other characteristics such as income, profession etc.

When the variation in the level of a single item is being studied, the index number is termed . as univariate index. But when the changes in average level of the number of items are being studied then collectively this index number is termed as composite index number. Most index numbers are composite in nature.

7.1 USES OF INDEX NUMBER

(1) ***Index Numbers are the economic barometers*** - According to G. Simpson & F. Kafta, "Index numbers are one of the most widely used statistical devices..... They are used to take the pulse of the economy and they have come to be used as indicators of inflationary or deflationary tendencies"

A barometer is an instrument that is used to measure atmospheric pressure. Index numbers are used to feel the pressure of the economic and business behaviour, as well as to measure the change in general economic conditions of a country. Index numbers are indispensable tools in planning and control and both for government organisations and for individual business concerns.

(2) ***Index number helps in formulation of policy decisions*** - Index number relating to output (industrial production, agricultural production),volume of imports and export, volume of trade, foreign exchange reserve and other financial matters are indispensable for any government organisation as well as private business concerns in efficient planning and formulating policy decisions.

(3) ***Index numbers reveal trends and tendencies*** - Index numbers reflect the pattern of change in the level of a phenomenon. For example, by examining the index number for imports and export for the last 10 years, we can draw the trend of the phenomenon under study and can also draw conclusions.

(4) ***Index numbers help to measures the Purchasing Power of money*** - Once the price index is computed, then the earnings of a group of people or class is adjusted with a price index that provides an overall view of the purchasing power for the group.

(5) ***Consumer price indices are used for deflating*** - The price index number is useful in deflating the national income to remove the effect of inflation over a long term, so that we may understand whether there is any change in the real income to the people or not.

7.2 PROBLEMS INVOLVED IN CONSTRUCTION OF INDEX NUMBERS

There are several problems that a statistician encounter in process of construction of Index Numbers. These are as follows—

(1) ***A clear definition of the purpose for which the index is constructed should be made***. Before collection of data for construction of index numbers, it is of utmost importance to know what is the purpose for construction of index numbers. For example if we wish to measure trend in price changes with a view point of finding the consumption pattern of a household; in such a case we should take retail prices and not wholesale prices of items into consideration.

(2) ***Selection of number of items***. Those items which are relevant for a partiality type of changes are to be selected, for example in computing the cost of living index Number of a middle class family gold will not be a relevant item, where as family clothing should be included.

(3) ***Base period*** - Base period is a reference period whose level of prices (in case of Price Index) represents the base from which changes in prices are measured. For example when we compare the prices of wheat in the year 2008 with that of 2000, the year 2000 is the base year.

The choice of base period is very critical in construction of Index Numbers and it is based on the following two considerations- *(a)* base year should be a normal period *i.e.* period with relative stability and should not be affected by extraordinary events like war, famine etc. *(b)* It should not be in too distant past.

The choice is also to be made about the kind of base to be used *i.e.* whether fixed base should be used or chain base should be used.

(4) ***Selection of weights*** - Weights imply the relative importance of the different variables. It is very essential to adopt a suitable method of weighting to avoid arbitrary & haphazard weights. For instance, in computing cost of living index, wheat or rice should be given more importance as compared to sugar or salt.

(5) ***Adoption of suitable formula for construction of index number***- As there are number of formulas to calculate index number ; most appropriate & proper one should be used & selected depending upon the circumstances.

7.3 METHOD OF CONSTRUCTION OF INDEX NUMBERS

Index numbers may be constructed by any of the following methods—

(1) Unweighted Index :

(a) Simple Aggregative Index

(b) Simple Average of Relatives

(2) Weighted Indices:

(a) Weighted Aggregative. Index

(b) Weighted Average of Relatives

- Index Numbers
  - Unweighted
    - Simple aggregate
    - Simple average of price relative
  - Weighted
    - Weighted aggregate
    - Weighted average of price relatives

7.3.1 UNWEIGHTED INDEX : SIMPLE AVERAGE OF PRICE RELATIVE METHOD

Under this method the price of each commodity in the current year is taken as a percentage of the price of corresponding item of the base year and the index is obtained by averaging these percentage figures. Arithmetic mean or geometric mean may be used to average these percentages.

When arithmetic mean is used for averaging the relatives, the formula for computing the index is :

$$P_{01} = \frac{\frac{p_1}{p_0} \times 100}{N}$$

Where p_1 = price of current year

p_0 = price of base year

N = Total Number of items

When geometric mean is used for averaging the relatives, the formula for computing the index is :

$$P_{01} = \text{Antilog}\ \frac{\log \frac{p_1}{p_0} \times 100}{N}$$

Example 1 :

From the following data construct an index for 2012 taking 2011 as base by Price Relative method using

(a) Arithmetic Mean

(b) Geometric Mean

for averaging relatives:

Commodities	*Price in 2011*	*Price in 2012*
A	8	12
B	6	8
C	5	6
D	48	52
E	15	18
F	9	27

Solution :

Index Number using Arithmetic Mean of Price Relative

Table : Calculation of Price Relatives of Arithmetic Mean

Commodities	*Prices in 2011* P_0	*Prices in 2012* P_1	*Price Relatives* $P = \frac{p_1}{p_0} \times 100$
A	8	12	150
B	6	8	133.33
C	5	6	120
D	48	52	108.3
E	15	18	120
F	9	27	300
			$\Sigma P = 931.63$

$$P_{01} = \frac{\Sigma p}{N} = \frac{931.63}{6} = 155.27\%$$

To determine the value of *a* and b, the following two normal equations are to be solved simultaneously:

Table : Calculation of Price Relatives of Geometric Mean

Commodities	p_0	p_1	$p = \frac{p_1}{p_0} \times 100$	*log P*
A	8	12	150	2.1761
B	6	8	133.33	2.1249
C	5	6	120	2.0792
D	48	52	108.3	2.0346
E	15	18	120	2.0792
F	9	27	300	2.4771
				$\Sigma \log P = 12.9711$

$$P_{01} = \text{Antilog } \frac{12.9711}{6}$$

$$= \text{Antilog } 2.1618$$

$$= 145.1\%$$

Note : *The difference in the answer is due to the method of averaging used.*

Unweighted Indices

(a) ***Simple Aggregate Method -*** This is the simplest method of construction index numbers. It consists of expressing the aggreate price of all commodities in the current year as a per cent of the aggregate price in the base year. Symbolically:

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

Where

Σp_0 = total of prices of all commodities of base year

Σp_1 = total of prices of all commodities of current year

p_{01} = Index Number of current year

Examples : 2

From the following data construct a price Index for year 2012 taking year 2009 as base

Commodities	*Prices in 2009*	*Prices in 2012*
Potato (per Kg)	12	20
Wheat (per Kg)	20	25
Bread	10	13
Chese (per 100 gms)	8	10

Solution :

Table : Construction of Price Index

Commodities	P_0	P_1
Potato (per Kg)	12	20
Wheat (per Kg)	20	25
Bread	10	13
Chese (per 100 gms)	8	10
	$\Sigma p_0 = 50$	$\Sigma p_1 = 68$

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

$$= \frac{68}{50} \times 100$$

$$P_{01} = 136$$

This means that as compared to 2009, in 2012 there is a net increase in the prices of the given commodities to the extent of 36%.

This method has following limitations –

(1) The index is affected by the units in which the prices are quoted (such as litres, kilogram etc.). In the preceding example, if the prices of chese is taken in per kg (instead of as per 100 gms) *e.g.* ₹ 80 in 2009 and ₹ 100 in 2012 the index so computed would differ as follows :

Table : Construction of Price Index

Commodities	P_0	P_1
Potato (per Kg)	12	20
Wheat (per Kg)	20	25
Bread	10	13
Cheese (per Kg)	80	100
	$\Sigma p_0 = 122$	$\Sigma p_1 = 158$

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

$$= \frac{158}{122} \times 100$$

$$P_{01} = 129$$

The net increase in price now is only 29%.

(2) The relative importance of various commodities is not taken into account in as it is unweighted. Thus according to this method equal weights (importance) would be attached to wheat and salt in computing a cost of living index.

(3) This method is influenced by the magnitude of prices i.e. the higher the price of the commodity, the greater is the influence on the Index number. Such price quotations become the concealed weights which have no logical significance.

7.3.2 Weighted Aggregate Method

In this method, appropriate weights are assigned to various commodities to reflect their relative importance in group. For the construction of price index number, quantity, weights are used *i.e.* amount of quantity consumed, purchased or marketed.

By using different systems of weighting we get a number of formulae which are as follows—

Laspeyres' Price Index

In this method the base year quantities are taken as weights. Symbolically—

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

The main advantage of this method is that it uses only base year quantity ; therefore there is no need to keep record of quantity consumed in each year.

Disadvantage

It is a common knowledge that the consumption of commodity decreases with relative large increase in price and *vice versa.* Since in this method base year quantity is taken as weights, it does not take into account the change in consumption due to increase or decrease in prices and hence may give a biased result.

Paasche's Method

In this method current year quantities are taken as weights. Symbolically—

$$P_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

Dorbish and Bowley's Method

This method is the simple arithmetic mean of the Laspeyres' and Paasche's indices. This index takes into account the influence of quantity weights of both base period and current period. The formula is as follows:

$$P_{01} = \frac{\dfrac{\Sigma p_1 q_0 + \Sigma p_1 q_1}{\Sigma p_0 q_0 + \Sigma p_0 q_1}}{2} \times 100$$

Fisher 'Ideal' Method

This method is the geometric mean of Laspeyres' and Paasche's indices. The formula is as follows—

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

Advantages

Because of the following advantages this method is seldom referred as ideal method—

(1) The formula takes into account both base year and current year quantities as weights, and hence avoids bias associated with the Laspeyres' and Paasche's indices.

(2) The formula is based on geometric mean which is considered to be the best average for constructing index numbers.

(3) This method satisfies unit test, time reversal test and factor reversal test.

Disadvantage

(1) This method is more time consuming than other methods.

(2) It also does not satisfy circular test.

Marshall-Edgeworth Method

In this method arithmetic mean of base year and current year quantities are taken as weights symbolically—

$$P_{01} = \frac{\Sigma p_1 \dfrac{q_1 + q_2}{2}}{\Sigma p_0 \dfrac{q_1 + q_2}{2}} \times 100$$

Kelly's Method

In this method fixed weights are taken as weights. This method is sometimes referred to as *aggregative index with fixed weights method*. Fixed weights are quantities which may be for some particular period (not necessarily of base year or the current year) and this is kept constant all the time. The formula is as follows—

$$P_{01} = \frac{\Sigma p_1 q}{\Sigma p_0 q} \times 100$$

Advantage

(1) This index does not require yearly changes in the weights.

Disadvantage

This method does not take into account the weight of either the base year or the current year.

Example 3 :

Compute the price index as per the following methods :

(1) Laspeyres'
(2) Paasche's
(3) Fisher
(4) Bowley's
(5) Marshall - Edgeworths

from the following :

Item	p_0	q_0	p_1	q_1
A	10	4	12	6
B	15	6	20	4
C	2	5	5	3
D	4	4	4	4

Solution :

Table : Calculation of various indices

	p_0	q_0	p_1	q_1	p_0q_0	p_0q_1	p_1q_0	p_1q_1
A	10	4	12	6	40	60	48	72
B	15	6	20	4	90	60	120	80
C	2	5	5	3	10	6	25	15
D	4	4	4	4	16	16	16	16
					$\Sigma p_0q_0=156$	$\Sigma p_0q_1=142$	$\Sigma p_1q_0=209$	$\Sigma p_1q_1=183$

(i) Laspeyres' Index Number

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

$$= \frac{209}{156} \times 100 = 133.97$$

(ii) Paasche's Index Number

$$P_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

$$= \frac{183}{142} \times 100 = 128.87$$

(iii) Bowley's Index Number

$$P_{01} = \frac{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}{2} \times 100$$

$$= \frac{\frac{209}{156} + \frac{183}{142}}{2} \times 100$$

$$= \frac{1.34 + 1.29}{2} \times 100$$

$$= \frac{2.63}{2} \times 100 = 131.5$$

(iv) Fisher's Index Number

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

$$= \sqrt{\frac{209}{156} \times \frac{183}{142}} \times 100$$

$$= \sqrt{\frac{38247}{22152}} \times 100$$

$$= \sqrt{1 \cdot 726} \times 100 = 1 \cdot 31 \times 100$$

$$= 131$$

(v) Marshall-Edgeworth Index Number

$$P_{01} = \frac{\Sigma p_1 (q_0 + q_1)}{\Sigma p_0 (q_0 + q_1)} \times 100$$

$$= \frac{\Sigma p_1 q_0 + \Sigma p_1 q_1}{\Sigma p_0 q_0 + \Sigma p_0 q_1} \times 100$$

$$= \frac{209 + 183}{156 + 142} \times 100$$

$$= \frac{382}{298} \times 100 = 131.54$$

7.3.3 Weighted Index : Weighted Average of Relative Method

In this method price of each commodity in the current year is taken as a percentage of the price of corresponding item of the base year. These relatives are multiplied by the given weights and the result is obtained by averaging the resulting figures. Arithmetic mean or geometric mean is used to average these figures.

When Arithmetic Mean is used for averaging weighted relatives the formula is :

$$P_{01} = \frac{\Sigma PV}{\Sigma V}$$

where $P = \frac{p_1}{p_0} \times 100$

$V = p_0 q_0$

When Geometric Mean is used for averaging weighted relatives the formula is :

$$P_{01} = \text{antilog} \frac{\Sigma(\log P \times V)}{\Sigma V} \times 100$$

where $P = \frac{p_1}{p_0} \times 100$

$V = p_0 q_0$

Example 4 :

Compute price index by Weighted Average of Relatives using—

(1) Arithmetic mean

(2) Geometric mean

Items	*Price in 2011*	*Price in 2012*	*Quantity in 2011*
A	5	6	2
B	4	5	0.25
C	15	16.5	1

Solution :

Table : Computation of price index using arithmetic mean

Items	p_0	p_1	q_0	$P = \frac{p_1}{p_0} \times 100$	$V = p_0 q_0$	*PV*
A	5	6	2	$\frac{6}{5} \times 100 = 120$	10	1200
B	4	5	0.25	$\frac{5}{4} \times 100 = 125$	1	125
C	15	16.5	1	$\frac{16.5}{15} \times 100 = 110$	15	1650
					$\Sigma V = 26$	$\Sigma PV = 2975$

$$P_{01} = \frac{\Sigma PV}{\Sigma V}$$

$$= \frac{2975}{26}$$

$P_{01} = 114.42$

Computation of Price Index using geometric mean

Items	p_0	p_1	q_0	$P = \frac{p_1}{p_0} \times 100$	$V = p_0q_0$	*logP*	*V x log P*
A	5	6	2	120	10	2.0792	20.792
B	4	5	0.25	125	1	2.0969	2.0969
C	15	16.5	1	110	15	2.0414	30.621
					$\Sigma V = 26$		$\Sigma \log PV = 53.5099$

$$P_{01} = \text{antilog} \frac{\Sigma \log P \times V}{\Sigma V}$$

$$= \text{antilog} \frac{53.5099}{26}$$

$$= \text{antilog } 2.0581$$

$$= 114.3$$

Example 5 :

The following table gives the prices of some food items in the base year & current year & the quantities sold in the base year. Calculate the weighted index number by using the Weighted Average of Price Relatives :

Items	*Base year quantities (units)*	*Base year Price (in ₹)*	*Current year Price (in ₹)*
A	7	18.00	21
B	6	3.00	4
C	16	7.50	9
D	21	2.50	2.25

Solution :

Table : Calculation of Weighted Average of Price Relation

Item	*Base year quantities*	*Base year Price*	*Current Price*	$P = \frac{p_1}{p_0} \times 100$	$V = p_0q_0$	*PV*
	q_0	p_0	p_1			
A	7	18	21	$\frac{21}{18}$ x 100 = 116.66	18 x 7 =126	14699.16
B	6	3	4	$\frac{4}{3}$ x 100 =133.33	6 x 3 =18	2399.94
C	16	7.5	9	$\frac{9}{7.5}$ x 100 =120	16 x 7.5 =120	14400
D	21	2.5	2.25	$\frac{2.25}{2.5}$ x 100 = 90	21X2.5 = 52.5	4725
					$\Sigma V = 316.5$	ΣPV=36224.1

$$P_{01} = \frac{\Sigma PV}{\Sigma V}$$

$$= \frac{36224.1}{316.5} = 114.45$$

7.4 QUANTITY INDEX NUMBERS

Just as the price index number measures the changing prices of the goods so a quantity index number measures the change in quantity/volume of the goods produced, sold or consumed. The method of construction of quantity index number are similar to the methods discussed above in the context of price index. The only difference is that the quantity index formula are obtained from the corresponding price index formula by an interchange of *p* by *q* & *q* by *p*.

Thus the following list of formulae can be derived :

Unweighted Index : Simple Aggregative Method

$$Q_{01} = \frac{\Sigma q_1}{\Sigma q_0} \times 100$$

Unweighted Index : Simple Average of Quantity Relative Method

– When Arithmetic Mean is used for averaging the relatives

$$Q_{01} = \frac{\Sigma \frac{q_1}{q_0} \times 100}{N}$$

– When Geometric Mean is used for averaging the relatives

$$Q_{01} = \text{antilog} \frac{\Sigma \log \frac{q_1}{q_0} \times 100}{N}$$

Weighted Index : Simple Aggregative Method

– Laspeyres' Method

$$Q_{01} = \frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times 100$$

– Paasche's Method

$$Q_{01} = \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1} \times 100$$

– Dorbish & Bowley's Method

$$Q_{01} = \frac{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} + \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}}{2} \times 100$$

– Fisher 'ideal' Method

$$Q_{01} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}} \times 100$$

– Marshall-Edgeworth Method

$$Q_{01} = \frac{\Sigma q_1 \frac{p_1 + p_2}{2}}{\Sigma q_0 \frac{p_1 + p_2}{2}} \times 100$$

— Kelly's Method

$$Q_{01} = \frac{\Sigma q_1 p}{\Sigma q_0 p} \times 100$$

Weighted Index : Weighted Average of Relative Method

– When Arithmetic Mean is used for averaging

$$Q_{01} = \frac{\Sigma QV}{\Sigma V}$$

where $Q = \frac{q_1}{q_0} \times 100$ & $V = q_0 p_0$

– When Geometric Mean is used for averaging

$$Q_{01} = \text{antilog} \frac{(\log Q \times V)}{\Sigma V}$$

7.5 VALUE INDEX NUMBER

The value of a commodity is the product of its price and quantity. Thus the value of index is the sum of the values of a given period ($\Sigma p_1 q_1$) the base period ($\Sigma p_0 q_0$) The formula is :

$$V_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \times 100$$

Note: The weights are not to be applied in this case as they are interest in the value figures.

7.6 CONSUMER PRICE INDEX

The consumer price index measures the amount of money which consumer of a particular class have to pay to get a basket of goods & services at a particular point of time in comparison to what they paid for the same in the base period.

Different classes of people consume different types of commodities & even that same type of commodities are not consumed in the same proportion by different classes of people (for e.g. higher class, middle class, lower class). The general indices do not highlight the effects of change in prices of a various commodities consumed by different classes of people on their cost of living.

The consumer price index is also known as cost of living index or retail price index.

Methods of Constructing Consumer Price Index

The consumer price index can be constructed by any of the following two methods :

(1) Aggregate Expenditure Method or Aggregative Method

(2) Family Budget Method or the Method of Weighted Relatives

7.7 AGGREGATE EXPENDITURE METHOD

This method is similar to the Laspeyres' method of constructing a weighted index. In this method the quantities of various commodities consumed in the base year by a particular class of people are taken as weights.

$$\text{Consumer Price Index} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

Where p_1 & p_0 are prices of current year & base year respectively.

q_0 = quantity consumed in base year.

Family Budget Method

This method is same as the weighted average of price relative method discussed earlier. The formula is as follows :

$$\text{Consumer Price Index} = \frac{\Sigma PV}{\Sigma V}$$

$$\text{Where } P = \frac{p_1}{p_0} \times 100$$

V = Value weights *i.e.* $p_0 q_0$.

Uses of Consumer Price Index Number

(1) It is used to formulate economic policy and also to measure real earning.

(2) It is used to measure purchasing power of the consumer. The formula is as follows—

$$\text{Purchasing power} = \frac{1}{\text{Consumer Price Index}} \times 100$$

(3) It is used in deflating. The process of deflating can be expressed in the form of formula as—

$$\text{Real wage} = \frac{\text{Money Value}}{\text{Consumer Price Index}} \times 100$$

(4) It is used in wage negotiations & wage contracts. It also helps to calculate dearness allowance.

Example 6 :

An enquiry into the budgets of the middle class families in a city in India gave the following information :

	Food	*Rent*	*Clothing*	*Fuel*	*Others*
Expenses	35%	15%	20%	10%	20%
Price in 2011	150	50	100	20	60
Price in 2012	174	60	125	25	90

What change in the cost of living of 2012 has taken place as compared to 2011

Solution :

Table : Calculation of Cost of living

Items	*Expenses (%) W*	*Price in 2011 (p_0)*	*Price in 2012 (p_1)*	$p = \frac{p_1}{p_0} \times 100$	*PW*
Food	35	150	174	$\frac{174}{150} \times 100$	$\frac{174}{150} \times 100 \times 35 = 4060$
Rent	15	50	60	$\frac{60}{50} \times 100$	$\frac{60}{50} \times 100 \times 15 = 1800$
Clothing	20	100	125	$\frac{125}{100} \times 100$	$\frac{125}{100} \times 100 \times 20 = 2500$
Fuel	10	20	25	$\frac{25}{20} \times 100$	$\frac{25}{20} \times 100 \times 10 = 1250$
Others	20	60	90	$\frac{90}{60} \times 100$	$\frac{90}{60} \times 100 \times 20 = 3000$
	$\Sigma W = 100$				$\Sigma PW = 12610$

$$P_{01} = \frac{\Sigma PW}{\Sigma W} = \frac{12,610}{100} = 126.1$$

The cost of living in 2012 has increased by 26.1% as compared to 2011.

7.8 TEST OF ADEQUACY OF THE INDEX NUMBER FORMULAE

So far we have discussed various formulae for construction of weighted & unweighted index numbers. However the problem still remains of selecting an appropriate method for the construction of an index number in a given situation. The following tests can be applied to find out the adequacy of an index number.

(1) Unit Test

(2) Time Reversal Test

(3) Factor Reversal Test

(4) Circular Test

1. ***Unit Test*** - This test requires that the index number formulae should be independent of the units in which prices or quantities of various commodities are quoted. For example in a group of commodities, while the price of wheat might be in kgs., that of vegetable oil may be quoted in per liter & toilet soap may be per unit.

   Except for the simple (unweighted) aggregative index, all other formulae discussed above satisfy this test.

2. ***Time Reversal Test*** - The time reversal test is used to test whether a given method will work both backwards & forwards with respect to time. The test is that the formula should give the same ratio between one point of comparison & another no matter which of the two is taken as base.

   The time reversal test may be stated more precisely as follows—

   If the time subscripts of a price (or quantity) index number formula be interchanged, the resulting price (or quantity) formula should be reciprocal of the original formula.

   i.e. if p_0 represents price of year 2011 and p_1 represents price at year 2012 *i.e.*

   $\frac{p_1}{p_0}$ should be equal to $\frac{1}{p_0 / p_1}$

   symbolically, the following relation should be satisfied

   $p_{01} \times p_{10} = 1$, Omitting the factor 100 from both the indices.

   Where P_{01} is index for current year '1' based on base year '0'

   p_{10} is index for year '0' based on year '1'.

   The methods which satisfy the following test are:-

   (1) Simple aggregate index
   (2) Simple geometric mean of price relative
   (3) Weighted geometric mean of price relative with fixed weights
   (4) Kelly's fixed weight formula
   (5) Fisher's ideal formula
   (6) Marshall-Edgeworth formula

   This test is not satisfied by Laspeyres' method & the Paasche's method as can be seen from below—

   $$\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \neq 1 \text{ (Laspeyres' Method)}$$

   Similarly when Paasche method is used—

   $$\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0} \neq 1$$

   On other hand applying Fisher's formula

   $$p_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \quad \text{(Omitting the factor 100)}$$

   $$\text{and } p_{10} = \sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}} \quad \text{(Omitting the factor 100)}$$

$$p_{01} \times p_{10} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times \frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_1 q_0}} = \sqrt{1} = 1$$

Hence the test is satisfied.

3. ***Factor Reversal Test*** - An Index number formula satiesfies this test if the product of the Price Index and the Quantity Index gives the **True value ratio**, omitting the factor 100 from each index. This test is satisfied if the change in the price multiplied by the change in quantity is equal to the change in the value.

   Speaking precisely if p and q factors in a price (or quantity) index formula be interchanged, so that a quantity (or price) index formula is obtained the product of the two indices should give the true value ratio.

   Symbolically,

   $$P_{01} \times Q_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} = \text{The True Value Ratio} = \text{TVR}$$

   Consider the Laspeyres formula of price index

   $$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0}$$

   Consider the quantity index by interchange p with q & q with p

   $$Q_{01} = \frac{\Sigma q_1 p_0}{\Sigma q_0 p_0}$$

   $$\text{Now } P_{01} \times Q_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \neq \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0}$$

   This test is not met.

   This test is only met by Fisher's ideal index. No other index number satisfies this test:

   Proof :

   $$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}$$

   Changing p to q and q to p, we get

   $$Q_{01} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}}$$

   $$P_{01} \times Q_{01} = \sqrt{\frac{\cancel{\Sigma p_1 q_0}}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\cancel{\Sigma p_0 q_1}} \times \frac{\cancel{\Sigma q_1 p_0}}{\Sigma q_0 p_1} \times \frac{\Sigma q_1 p_1}{\cancel{\Sigma q_0 p_1}}}$$

   $$= \sqrt{\frac{(\Sigma q_1 p_1)^2}{(\Sigma q_0 p_0)^2}} = \frac{\Sigma q_1 p_1}{\Sigma q_0 p_0} = \text{True Value Ratio}$$

4. ***Circular Test*** - Circular test is an extension of time reversal test for more than two periods & is based on shiftability of the base period. For example, if an index is constructed for the year 2012 with the base of

2011 & another index for 2011 with the base of 2010. Then it should be possible for us to directly get an index for the year 2012 with the base of 2010. If the index calculated directly does not give an inconsistent value, the circular test is said to be satisfied.

This test is satisfied if—

$P_{01} \times P_{12} \times P_{20} = 1.$

This test is satisfied only by the following index Nos. formulas—

(1) Simple aggregative index

(2) Simple geometric mean of price relatives

(3) Kelly's fixed base method

When the test is applied to simple aggregative method—

$\therefore P_{01} \times P_{12} \times P_{20} =$

$$\frac{\Sigma p_1}{\Sigma p_0} \times \frac{\Sigma p_2}{\Sigma p_1} \times \frac{\Sigma p_0}{\Sigma p_2} = 1$$

Hence, the simple aggregative formula satisfies circular test

Similarly when it is applied to fixed weight Kelly's method

$$\frac{\Sigma p_1 q}{\Sigma p_0 q} \times \frac{\Sigma p_2 q}{\Sigma p_1 q} \times \frac{\Sigma p_0 q}{\Sigma p_2 q} = 1$$

This test is not satisfied by Fishers ideal index.

Example 7 :

Compute Fisher's Ideal Index and show that it satisfies time Reversal Test

	2009		*2012*	
Items	*Price (₹)*	*Value (₹)*	*Price (₹)*	*Value (₹)*
A	10	30	12	48
B	15	60	15	75
C	5	50	8	96
D	2	10	3	15

Solution :

As in this problem value of each item is given we have to find out quantity by dividing the value by the price. Symbolically :

Value = Price x Quantity

$$\therefore \text{Quantity} = \frac{\text{Value}}{\text{Price}}$$

Table : Calculation of Fisher's Ideal Index

	2009		2012					
Item	Price (₹)	Value (₹)	Price (₹)	Value (₹)	$q_0 = \frac{p_0q_0}{p_0}$	$q_1 = \frac{p_1q_1}{p_1}$	p_0q_1	p_1q_0
	p_0	p_0q_0	p_1	p_1q_1				
A	10	30	12	48	3	4	40	36
B	15	60	15	75	4	5	75	60
C	5	50	8	96	10	12	60	80
D	2	10	3	15	5	5	10	15
		$\Sigma p_0q_0=150$		$\Sigma p_1q_1=234$			$\Sigma p_0q_1=185$	$\Sigma p_1q_0=191$

$$P_{01} = \sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \frac{\Sigma p_1q_1}{\Sigma p_0q_1}} \times 100$$

$$= \sqrt{\frac{191}{150} \times \frac{234}{185}} \times 100 = \sqrt{1.273 \times 1.265} \times 100$$

$$= \sqrt{1.6039} \times 100$$

$$= 1.2691 \times 100 = 126.91$$

Time reversal test is satisfied when

$P_{01} \times P_{10} = 1$

$$P_{01} \times P_{10} = \sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times \frac{\Sigma p_1q_1}{\Sigma p_0q_1}} \times \sqrt{\frac{\Sigma p_0q_1}{\Sigma p_1q_1} \times \frac{\Sigma p_0q_0}{\Sigma p_1q_0}}$$

$$= \sqrt{\frac{191}{150} \times \frac{234}{185}} \times \sqrt{\frac{185}{234} \times \frac{150}{191}}$$

$$= \sqrt{\frac{191}{150} \times \frac{234}{185} \times \frac{185}{234} \times \frac{150}{191}}$$

$$= \sqrt{1} = 1$$

Since $P_{01} \times P_{10} = 1$ hence Fisher's ideal index satisfies time Reversal Test.

7.9 CHAIN INDEX NUMBERS

In the fixed base method which is discussed so far the base remains the same & does not change whole throughout the series. But with the passage of time some items may have been included in the series & other ones might have been deleted, & hence it becomes difficult to compare the result of present conditions with those of the old remote period. Hence the fixed base method does not suit when the conditions change. In such a case the changing base period may be more suitable. Under this method the figures for each year are first expressed as a percentage of the preceding year (called link relatives) then they are chained together by successive multiplication to form a chain index.

7.10. STEPS IN CONSTRUCTION OF CHAIN INDEX

(1) The figures are to be expressed as the percentage of the preceding year to get link relatives.

$$\text{Link Relatives of current year} = \frac{\text{price of current year}}{\text{price of previous year}} \times 100$$

(2) Chain index is obtained by the formula :

$$\text{Chain Index} = \frac{\text{Current year link relative} \times \text{Previous year link relative}}{100}$$

Example 8: From the following data find the index numbers by taking (1) 2005 as base *(ii)* by chain base method.

Year	2005	2006	2007	2008	2009	2010	2011	2012
Prices	60	62	65	72	75	80	82	85

Solution

Table : Construction of Index Number taking 2005 as base

Year	*Price*	*Index No. (2005 = 100)*
2005	60	100
2006	62	$\frac{62}{60}$ x 100 = 103.33
2007	65	$\frac{65}{60}$ x 100 = 108.33
2008	72	$\frac{72}{60}$ x 100 = 120
2009	75	$\frac{75}{60}$ x 100 = 125
2010	80	$\frac{80}{60}$ x 100 = 133
2011	82	$\frac{82}{60}$ x 100 = 136.66
2012	85	$\frac{85}{60}$ x 100 = 141.66

This means that from 2005 to 2006 there is an increase of 3.33% (103.33 - 100) from 2005 to 2007 there is an increase of 8.33% (108.33 - 100) increase from 2005 to 2008 there is an increase of 20% (120 - 100) & so on ....

Table : Calculation of Chain Base Index

Year	*Price*	*Linke Relative*	*Chain Indices (2005 = 100)*
2005	60	100	100
2006	62	$\frac{62}{60}$ x 100 = 103.33	103.33
2007	65	$\frac{65}{62}$ x 100 = 104.83	$\frac{104.83 \times 103.33}{100} = 108.32$
2008	72	$\frac{72}{65}$ x 100 = 110.76	$\frac{110.76 \times 108.32}{100} = 119.98$
2009	75	$\frac{75}{72}$ x 100 = 104.16	$\frac{104.16 \times 119.98}{100} = 124.97$
2010	80	$\frac{80}{75}$ x 100 = 106.66	$\frac{106.66 \times 124.97}{100} = 133.29$
2011	82	$\frac{82}{80}$ x 100 = 102.5	$\frac{102.5 \times 133.29}{100} = 136.62$
2012	85	$\frac{85}{82}$ x 100 = 103.65	$\frac{103.65 \times 136.62}{100} = 141.61$

Note: *The results obtained by the fixed base & chain base are almost similar. The difference is only due to approximation. Infact the chain base index numbers are always equal to fixed base index numbers if there is only one series.*

Example 9 :

From the following data compute chain base index number & fixed base index number.

Group	*2008*	*2009*	*2010*	*2011*	*2012*
I	6	9	12	15	18
II	8	10	12	15	18
III	12	15	24	30	36

Solution :

Table : Computation of Chain Base Index Number
Calculation of Link Relatives

Group	*2008*	*2009*	*2010*	*2011*	*2012*
I	100	$\frac{9}{6}$ x100 =150	$\frac{12}{9}$ x 100 =133.33	$\frac{15}{12}$ x 100 =125	$\frac{18}{15}$ x 100 =120
II	100	$\frac{10}{8}$ x 100 =125	$\frac{12}{10}$ x 100 =120	$\frac{15}{12}$ x 100 =125	$\frac{18}{15}$ x 100 =120
III	100	$\frac{15}{12}$ x 100 =125	$\frac{24}{15}$ x 100 =160	$\frac{30}{24}$ x 100 =125	$\frac{36}{30}$ x 100 =120

Group	*2008*	*2009*	*2010*	*2011*	*2012*
Total of Link Relatives	300	400	413.33	375	360
Average of Link Relatives	100	133.33	137.77	125	120
Chain Index	100	$\frac{100 \times 133.33}{100}$ = 133.33	$\frac{133.33 \times 137.77}{100}$ = 183.69	$\frac{183.69 \times 125}{100}$ = 229.61	$\frac{229.69 \times 120}{100}$ = 275.53

Table : Computation of Fixed base Index No.

Group	*2008*	*2009*	*2010*	*2011*	*2012*
I	100	$\frac{9}{6}$ x 100 =150	$\frac{12}{6}$ x 100 =200	$\frac{15}{6}$ x 100 =250	$\frac{18}{6}$ x 100 = 300
II	100	$\frac{10}{8}$ x 100 =125	$\frac{12}{8}$ x 100=150	$\frac{15}{8}$ x 100 =187.5	$\frac{18}{8}$ x 100 = 225
IIII	100	$\frac{15}{12}$ x 100 =125	$\frac{24}{12}$ x 100 =200	$\frac{30}{12}$ x 100 =250	$\frac{36}{12}$ x 100 =300
Total	300	400	550	687.5	825
AV	100	133.33	183.33	229.17	275

Note : *The two series of index numbers obtained by fixed base & chain base method are different except in the first two years. This would always be so in case of more than one series.*

Example 10 :

The price index & quantity index of a commodity were 120 & 110 respectively in 2012 with base 2011. Find its value index number in 2012 with base 2011.

Solution :

$$\frac{p_1}{p_0} \times 100 = 120 \qquad \frac{p_1}{p_0} \qquad 1.2$$

$$\frac{q_1}{q_0} \times 100 = 110 \qquad \frac{q_1}{q_0} \qquad 1.1$$

$$\therefore \text{Value index} = \frac{p_1 q_1}{p_0 q_0} \times 100$$

$$= 1.2 \times 1.1 \times 100 = 132$$

SELF EXAMINATION QUESTION

Problem I. Hence, factor reversal test is satisfied.

Problem 2. Explain the usefulness of constructing chain indices

Problem 3. Define Index Number. Explain the problems while constructing the Index No.

Problem 4. Discuss the problems faced while constructing an Index No.

Problem 5. Distinguish between fixed base Index and chain base Index Number.

Problem 6. Discuss the importance of cost of living index. What are the problems in construction of cost of living Index?

Problem 7. Distinguish between Laspeyres' and Paasche Index.

Unsolved Problems (Practical)

Problem I. Calculate Paasche's Quantity Index and Laspeyre's Price Index for the following data—

Commodities	*Quantity (Units)*		*Value (₹)*	
	2008	*2012*	*2008*	2012
A	100	150	500	900
B	80	100	320	500
C	60	72	120	360
D	30	33	360	297

(**Ans.** 131.2, 120.77)

Problem 2. Calculate Fisher's Ideal Index from the following data and show that it satisfies Time Reversal Test.

Commodity	*2010-2011*		*2011-2012*	
	Price	*Quantity*	*Price*	*Quantity*
A	10	100	12	96
B	8	96	8	104
C	12	144	5	120
D	20	300	25	250
E	5	40	8	64
F	2	20	4	24

(**Ans.** 109.26)

Problem 3. From the fixed base Index Numbers given below find out chain base index number—

Year	2008	2009	2010	2011	2012
Fixed base index numbers	267	275	280	290	320

(**Ans.** 100, 103, 101, 103, 110.4)

Problem 4. From the chain base index numbers given below prepare fixed base index numbers—

Year	2008	2009	2010	2011	2012
Chainbase Index No.	80	110	120	105	95

(**Ans.** 80, 88, 105.6, 110.9, 105.33)

Problem 5. Compute Fisher's index number from the following data & show that it satisfies time reversal test and factor reversed test.

Commodity	*A*		*B*		*C*	
	Price	*Quantity*	*Price*	*Quantity*	*Price*	*Quantity*
Base year	4	50	3	10	2	5
Current year	10	40	8	8	4	4

(**Ans.** 250)

UNWEIGHTED INDEX

A. SIMPLE AVERAGE OF PRICE RELATIVE METHOD

- **When Arithmetic mean is used**

$$P_{01} = \frac{\Sigma \frac{p_1}{p_0} \times 100}{N}$$

- **When geometric mean is used**

$$P_{01} = \text{Antilog} \frac{\Sigma \log \frac{p_1}{p_0} \times 100}{N}$$

B. SIMPLE AGGREGATIVE METHOD

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

WEIGHTED INDEX

A. WEIGHTED AGGREGATIVE METHOD

- **Laspeyres' Method**

$$P_{01} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

- **Paasche's Method**

$$P_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

- **Dorbish and Bowley's Method**

$$P_{01} = \frac{\frac{\Sigma p_1 q_0 + \Sigma p_1 q_1}{\Sigma p_0 q_0 + \Sigma p_0 q_1}}{2} \times 100$$

- **Fisher's 'Ideal' Method**

$$P_{01} = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

- **Marshall - Edgeworth Method**

$$P_{01} = \frac{\Sigma p_1 \frac{q_1 + q_2}{2}}{\Sigma p_0 \frac{q_1 + q_2}{2}} \times 100$$

- **Kelly's Method**

$$P_{01} = \frac{\Sigma p_1 q}{\Sigma p_0 q} \times 100$$

B. WEIGHTED AVERAGE OF RELATIVE METHOD

- **When Arithmetic mean is used**

$$P_{01} = \frac{\Sigma PV}{\Sigma V}$$

- **When Geometric mean is used**

$$P_{01} = \text{antilog} \frac{\Sigma(\log P \times V)}{\Sigma V}$$

QUANTITY INDEX NUMBERS

A. UNWEIGHTED INDEX : SIMPLE AGGREGATIVE METHOD

$$Q_{01} = \frac{\Sigma q_1}{\Sigma q_0} \times 100$$

B. UNWEIGHTED INDEX : SIMPLE AVERAGE OF QUANTITY RELATIVE METHOD

When Arithmetic mean for averaging is used

$$Q_{01} = \frac{\Sigma \frac{q_1}{q_0} \times 100}{N}$$

When Geometric Mean is used for averaging the relatives

$$Q_{01} = \frac{\Sigma \log \frac{q_1}{q_0} \times 100}{N}$$

WEIGHTED INDEX

A. SIMPLE AGGREGATIVE METHOD

- **Laspeyres' Method**

$$Q_{01} = \frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times 100$$

- **Paasche's Method**

$$Q_{01} = \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1} \times 100$$

- **Dorbish & Bowley's Method**

$$Q_{01} = \frac{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} + \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}}{2} \times 100$$

- **Fisher's 'Ideal' Method**

$$Q_{01} = \sqrt{\frac{\Sigma q_1 p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_1 p_1}{\Sigma q_0 p_1}} \times 100$$

- **Marshall-Edgeworth Method**

$$Q_{01} = \frac{\Sigma q_1 \dfrac{p_1 + p_2}{2}}{\Sigma q_0 \dfrac{p_1 + p_2}{2}} \times 100$$

- **Kelly's Method**

$$Q_{01} = \frac{\Sigma q_1 p}{\Sigma q_0 p} \times 100$$

B. WEIGHTED AVERAGE OF RELATIVE METHOD

- **When Arithmetic mean is used for averaging**

$$Q_{01} = \frac{\Sigma QV}{\Sigma V}$$

where $Q = \frac{q_1}{q_0} \times 100$ & $V = q_0 p_0$

- **When Geometric mean is used for averaging**

$$Q_{01} = \text{antilog} \frac{(\log Q \times \upsilon)}{\Sigma \upsilon}$$

VALUE INDEX NUMBER

$$V_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \times 100$$

CONSUMER PRICE INDEX

- **Aggregate Expenditure Method**

$$\text{Consumer Price Index} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

- **Family Budget Method**

$$\text{Consumer Price Index} = \frac{\Sigma PV}{\Sigma V}$$

CHAIN INDEX NUMBERS

$$\text{Chain Index} = \frac{\text{Current year link relative} \times \text{Previous year chain index}}{100}$$

Study Note - 8

TIME SERIES ANALYSIS

This Study Note includes

Time series is statistical data that are arranged and presented in a chronological order i.e., over a period of time.

8.1 DEFINITION

According to Spiegel, "A time series is a set of observations taken at specified times, usually at equal intervals."

According to Ya-Lun-Chou, "A time series may be defined as a collection of reading belonging to different time period of same economic variable or composite of variables."

8.2 COMPONENTS OF TIME SERIES

There are various forces that affect the values of a phenomenon in a time series; these may be broadly divided into the following four categories, commonly known as the components of a time series.

(1) Long term movement or Secular Trend

(2) Seasonal variations

(3) Cyclical variations

(4) Random or irregular variations

(1) ***Secular Trend or Simple trend -*** The general tendency of a data to increase or decrease or stagnate over a long period of time is called secular trend or simple trend.

Most of the time series relating to Economic, Business and Commerce might show an upward tendency in case of population, production & sales of products, incomes, prices; or downward tendency might be noticed in time series relating to share prices, death, birth rate etc. due to global melt down, or improvement in medical facilities etc. All these indicate trend.

(2) ***Seasonal variations -*** Over a span of one year, seasonal variation takes place due to the rhythmic forces which operate in a regular and periodic manner. These forces have the same or almost similar pattern year after year.

Seasonal variations could be seen and calculated if the data are recorded quarterly, monthly, weekly, daily or hourly basis. So if in a time series data only annual figures are given, there will be no seasonal variations.

The seasonal variations may be due to various seasons or weather conditions for example sale of cold drink would go up in summers & go down in winters. These variations may be also due to man-made conventions & due to habits, customs or traditions. For example sales might go up during Diwali & Christmas or sales of restaurants & eateries might go down during Navratris.

(3) ***Cyclical variations*** - These variations in a time series are due to ups & downs recurring after a period from time to time. Though they are more or less regular, they may not be uniformly periodic. These are oscillatory movements which are present in any business activity and is termed as business cycle. It has got four phases consisting of prosperity (boom), recession, depression and recovery. All these phases together may last from 7 to 9 years may be less or more.

(4) ***Random or irregular variations*** - These fluctuations are a result of unforeseen and unpredictably forces which operate in absolutely random or erratic manner. They do not have any definite pattern and it cannot be predicted in advance. These variations are due to floods, wars, famines, earthquakes, strikes, lockouts, epidemics etc.

8.3 MODELS OF TIME SERIES ANALYSIS

The following are the two models which are generally used for decomposition of time series into its four components. The objective is to estimate and separate the four types of variations and to bring out the relative impact of each on the overall behaviour of the time series.

(1) Additive model

(2) Multiplicative model

Additive Model - In additive model it is assumed that the four components are independent of one another *i.e.* the pattern of occurrence and magnitude of movements in any particular component does not affect and are not affected by the other component. Under this assumption the four components are arithmetically additive *ie.* magnitude of time series is the sum of the separate influences of its four components *i.e.*

$$Y_t = T + C + S + I$$

Where

Y_t = Time series

T = Trend variation

C = Cyclical variation

S = Seasonal variation

C = Random or irregular variation

Multiplicative Model - In this model it is assumed that the forces that give rise to four types of variations are interdependent, so that overall pattern of variations in the time series is a combined result of the interaction of all the forces operating on the time series. Accordingly, time series are the product of its four components *i.e.*

$$Y_t = T \times C \times S \times I$$

As regards to the choice between the two models, it is generally the multiplication model which is used more frequently. As the forces responsible for one type of variation are also responsible for other type of variations, hence it is multiplication model which is more suited in most business & economic time series data for the purpose of decomposition.

8.4 MEASUREMENT OF SECULAR TREND

The following are the methods most commonly used for studying & measuring the trend component in a time series—

(1) Graphic or a Freehand Curve method

(2) Method of Semi Averages

(3) Method of Moving Averages

(4) Method of Least Squares

Graphic or Freehand Curve Method

The data of a given time series is plotted on a graph and all the points are joined together with a straight line. This curve would be irregular as it includes short run oscillation. These irregularities are smoothened out by drawing a free hand curve or line along with the curve previously drawn.

This curve would eliminate the short run oscillations & would show the long period general tendency of the data. While drawing this curve it should be kept in mind that the curve should be smooth and the number of points above the trend curve should be more or less equal to the number of points below it.

Merits

(1) It is very simple and easy to construct.

(2) It does not require any mathematical calculations and hence even a layman can understand it.

Disadvantages

(1) This is a subjective concept. Hence different persons may draw free hand lines at different positions and with different slopes.

(2) If the length of period for which the curve is drawn is very small, it might give totally erroneous results.

8.5 METHOD OF SEMI AVERAGES

Under this method the whole time series data is classified into two equal parts and the averages for each half are calculated. If the data is for even number of years, it is easily divided into two. If the data is for odd number of years, then the middle year of the time series is left and the two halves are constituted with the period on each side of the middle year.

The arithmetic mean for a half is taken to be representative of the value corresponding to the mid point of the time interval of that half. Thus we get two points. These two points are plotted on a graph and then are joined by straight line which is our required trend line.

8.6 MOVING AVERAGE METHOD

A moving average is an average (Arithmetic mean) of fixed number of items (known as periods) which moves through a series by dropping the first item of the previously averaged group and adding the next item in each successive average. The value so computed is considered the trend value for the unit of time falling at the centre of the period used in the calculation of the average.

In case the period is odd- If the period of moving average is odd for instance for computing 3 yearly moving average, the value of 1st, 2nd & 3rd years are added up and arithmetic mean is found out and the answer is placed against the 2nd year; then value of 2nd, 3rd & 4th years are added up & arithmetic mean is derived and this average is placed against 3rd year *(ie.* the middle of 2nd, 3rd & 4th) and so on.

In case of even number of years - If the period of moving average is even for instance for computing 4 yearly moving average, the value of 1st, 2nd, 3rd & 4th years are added up & arithmetic mean is found out and answer is placed against the middle of 2nd & 3rd year. The second average is placed against middle of 3rd & 4th year. As this would not coincide with a period of a given time series an attempt is made to synchronise them with the original data by taking a two period average of the moving averages and placing them in between the corresponding time periods. This technique is called centering & the corresponding moving averages are called moving average centred.

Example 1 :

The wages of certain factory workers are given as below. Using 3 yearly moving average indicate the trend in wages.

Year	2004	2005	2006	2007	2008	2009	2010	2011	2012
Wages	1200	1500	1400	1750	1800	1700	1600	1500	1750

Solution :

Table : Calculation of Trend Values by method of 3 yearly Moving Average

Year	*Wages*	*3 yearly moving totals*	*3 yearly moving average* i.e. *trend*
2004	1200	—	—
2005	1500	4100	1366.67
2006	1400	4650	1550
2007	1750	4950	1650
2008	1800	5250	1750
2009	1700	5100	1700
2010	1600	4800	1600
2011	1500	4850	1616.67
2012	1750	—	—

Example 2 :

Calculate 4 yearly moving average of the following data—

Year	2005	2006	2007	2008	2009	2010	2011	2012
Wages	1150	1250	1320	1400	1300	1320	1500	1700

Solution :

First Method :

Table : Calculation of 4 year Centered Moving Average

Year (1)	**Wages (2)**	**4 yearly moving total (3)**	**2 year moving total of col. 3 (centered) (4)**	**4 yearly moving average centred (5) [col. 4/8]**
2005	1150	–	–	–
2006	1250	–	–	–
		5,120		
2007	1320		10,390	1298.75
		5,270		
2008	1400		10,610	1326.25
		5,340		
2009	1300		10,860	1,357.50
		5,520		
2010	1320		11,340	1,417.50
		5,820		
2011	1500			
2012	1700			

Second Method :

Table : Calculation of 4 year Centered Moving Average

Year	**Wages**	**4 yearly moving total (3)**	**4 yearly moving average (4)**	**2 year moving total of col. 4 (centered) (5)**	**4 year centred moving average (col. 5/2)**
2005	1150	–	–	–	–
2006	1250	–	–	–	– –
		5,120	1,280	–	–
2007	1,320			2597.75	1298.75
		5,270	1317.5	–	–
2008	1,400			2,652.5	1,326.25
		5,340	1,335	–	–
2009	1,300			2,715	1,357.50
		5,520	1,380		
2010	1,320			2,835	1,417.50
		5,820	1,455	–	–
2011	1,500				
2012	1,700				

Example 3 :

Calculate five yearly moving averages for the following data—

Year	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
Value	123	140	110	98	104	133	95	105	150	135

Solution :

Table : Computation of Five Yearly Moving Averages

Year	*Value ('000 ₹)*	*5 yearly moving totals ('000 ₹)*	*5 yearly moving average ('000 ₹)*
2003	123	—	—
2004	140	—	—
2005	110	575	115
2006	98	585	117
2007	104	540	108
2008	133	535	107
2009	95	587	117.4
2010	105	618	123.6
2011	150	—	—
2012	135	—	—

8.7 METHOD OF LEAST SQUARES

The method of least squares as studied in regression analysis can be used to find the trend line of best fit to a time series data.

The regression trend line (Y) is defined by the following equation—

$Y = a + b X$

where Y = predicted value of the dependent variable

a = Y axis intercept or the height of the line above origin (*i.e.* when X = 0, Y = *a*)

b = slope of the regression line (it gives the rate of change in Y for a given change in X) (when *b* is positive the slope is upwards, when *b* is negative, the slope is downwards)

X = independent variable (which is time in this case)

To estimate the constants *a* and *b*, the following two equations have to be solved simultaneously—

$\Sigma Y = na + b\,\Sigma X$

$\Sigma XY = a\Sigma X + b\Sigma X^2$

To simplify the calculations, if the mid point of the time series is taken as origin, then the negative values in the first half of the series balance out the positive values in the second half so that Σx = 0. In this case the above two normal equations will be as follows—

$\Sigma Y = na$

$\Sigma XY = b\Sigma X^2$

In such a case the values of *a* and *b* can be calculated as under —

Since $\Sigma Y = na$,

$$a = \frac{\Sigma Y}{n}$$

Since $\Sigma XY = b\Sigma X^2$,

$$b = \frac{\Sigma XY}{\Sigma X^2}$$

Example 4 :

Fit a straight line trend to the following data by Least Square Method and estimate the sale for the year 2012 :

Year	2005	2006	2007	2008	2009	2010
Sale (in'000s)	70	80	96	100	95	114

Solution :

Table : Calculation of trend line

Year	*Sales* *Y*	*Deviations from 2007.5*	*Deviations multiplied by 2 (X)*	X^2	*XY*
2005	70	-2.5	-5	25	-350
2006	80	-1.5	-3	9	-240
2007	96	-.5	-1	1	-96
2008	100	+.5	+1	1	100
2009	95	+1.5	+3	9	285
2010	114	+2.5	+5	25	570
	$\Sigma Y = 555$			$\Sigma X^2 = 70$	$\Sigma XY = 269$

N = 6

Equation of the straight line trend is $Y_o = a + bX$

$$a = \frac{\Sigma Y}{N} = \frac{555}{6} = 92.5$$

$$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{269}{70} = 3.843$$

$\therefore$ Trend equation is $Y_c = 92.5 + 3.843X$

For 2012, X = 9

$Y_{2012} = 92.5 + 3.843 \times 9 = 92.5 + 34.587$

$= 126.59$ (in '000 ₹)

Example 5 :

Fit a straight line trend to the following data and estimate the likely profit for the year 2012. Also calculate the trend values.

Year	2003	2004	2005	2006	2007	2008	2009
Profit (in lakhs of ₹)	60	72	75	65	80	85	95

Solution :

Table : Calculation of Trend and Trend Values

Year	*Profit Y*	*Deviation from 2006*	X^2	*XY*	*Trend Values* $(Y_c = a + bX)$
		X			$[Y_c = 76 + 4.85X]$
2003	60	-3	9	-180	76 + 4.85 (-3) = 61.45
2004	70	-2	4	-144	76 + 4.85 (-2) = 66.30
2005	75	-1	1	-75	76 + 4.85 (-1) = 70.15
2006	65	0	0	0	76 + 4.85 (0) = 76
2007	80	1	1	80	76 + 4.85(1) = 80.85
2008	85	2	4	170	76 + 4.85 (2) = 85.70
2009	95	3	9	285	76 + 4.85 (3) = 90.55
	$\Sigma y = 532$		$\Sigma X^2 = 28$	$\Sigma XY = 136$	

N = 7

The equation for straight line trend is $Y_c = a + bX$

Where

$$\Sigma x = 0; \quad a = \frac{\Sigma Y}{N} = \frac{532}{7} = 76$$

$$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{136}{28} = 4.85$$

$$y_c = 76 + 4.85X$$

For 2012, x = 6 (2012 - 2006)

$y_c = 76 + 4.85(6) = 76 + 29.10$

= 105.10

The estimated profit for the year 2012 is ₹ 105.10 lakhs.

Example 6 :

Fit a straight line trend to the following data by Least Squares method and estimate exports for the year 2012.

Year	2003	2004	2005	2006	2007	2008	2009
Exports (in tons)	47	50	53	65	62	64	72

Solve by :

(1) taking 2005 as the year of origin

(2) taking middle year of the time series as origin and also verify the result.

Solution :

Table (1) Trend by taking 2005 as origin

Year	*Exports (in tons) (Y)*	*Deviation from 2005 X*	X^2	*XY*
2003	47	-2	4	-94
2004	50	-1	1	-50
2005	53	0	0	0
2006	65	1	1	65
2007	62	2	4	124
2008	64	3	9	192
2009	72	4	16	288
	$\Sigma Y = 413$	$\Sigma X = 07$	$\Sigma X^2 = 35$	$\Sigma XY = 525$

Equation of a straight line trend is.

$Y_c = a+bX$

To get the value of a and b, the following two normal equations have to be solved simultaneously—

$\Sigma Y = Na + b\Sigma X$

$\Sigma XY = a\Sigma X + b\Sigma X^2$

$413 = 7a + 7b$ ......(1)

$525 = 7a + 356$ ....(2)

Subtracting equation (1) from equation (2), we get

$112 = 28b$

$$b = \frac{112}{28} = 4$$

putting the value of b in equation 1

$413 = 7a + 7 \times 4$

$7a = 413 - 28$

$7a = 385$

$$a = \frac{385}{7} = 55$$

Equation for straight line trend is

$Y_c = 55 + 4X$ (origin 2005)

For 2005, X = 7

$Y_c = 55 + 4 \times 7 = 55 + 28 = 83$ tons

Table : Finding the trend line by taking middle year (2006) as origin

Year	*Exports (in tons)*	*Deviation from 2006 (X)*	X^2	*XY*
2003	47	-3	9	-141
2004	50	-2	4	-100
2005	53	-1	1	-53
2006	65	0	0	0
2007	62	1	1	62
2008	64	2	4	128
2009	72	3	9	216
	413	$\Sigma X = 0$	$\Sigma X^2 = 28$	$\Sigma XY=112$

The equation of the straight line trend is

$Y_c = a + bX$

$$\text{Since } \Sigma X = 0;\ a = \frac{\Sigma Y}{N} = \frac{413}{7} = 59$$

$$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{112}{28} = 4$$

The trend line is $Y_c = 59 + 4X$ (origin 2006)

For 2012 X = 6, so $Y_{2012} = 59 + 4(6) = 59 + 24$

Y = 83

The expected exports for the year 2012 are 83 tons.

Note : *We see that though trend line equations are different depending upon the year of origin, but the trend values would come out to be the same.*

Example 7 :

Fit the Straight Line Trend by method of least squares and estimate the sales for 2012.

Year	2006	2007	2008	2009	2010	2011
Sales	12	13	14	15	22	26

Solution :

Table : Calculation of Trend & Trend Values

Year	*Sales*	*Deviations from 2008.5 (X')*	*X = 2X'*	X^2	*XY*	$Y_c = a+bx$
2006	12	-2.5	-5	25	-60	17 + 1.4(-5) = 10
2007	13	-1.5	-3	9	-39	17 + 1.4(-3)= 12.8
2008	14	-0.5	-1	1	-14	17 + 1.4(-1)=15.6
2009	15	0.5	1	1	15	17 + 1.4(1) =18.4
2010	22	1.5	3	9	66	17 + 1.4(3) = 21.2
2011	26	2.5	5	25	130	17 + 1.4(5) = 24.0
	102	$\Sigma X' = 0$	$\Sigma X = 0$	$\Sigma X^2 = 70$	$\Sigma XY = 98$	

The equation of the straight line trend is

$Y_c = a + bX$

Since $\Sigma X = 0$; $a = \frac{\Sigma Y}{N} = \frac{102}{N} = 7$

$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{98}{70} = 1.4$

The trend line is $Y_c = 17 + 1.4X$

(ii) For 2012 X = *7*, so $Y_{2012} = 17 + 1.4(7) = 17 + 9.8$

Y = 26.8

*[**Note :** The deviations are taken from the middle year 2008.5 to reduce the calculations & then the resultant figures are multiplied by 2, to make calculations less cumbersome]*

Example 8 :

Below are given the figures of sales in thousand quintals of a firm operating in the sugar industry :

Year	2001	2003	2005	2007	2009
Sales in'000 quintals	70	90	100	130	170

(i) Fit straight line trend to these figures using the least squares method

(ii) Estimate the sales of the firm for the year 2012

(iii) What is the annual increase or decrease in the expected sales of the firm?

Solution :

(i) Table : Calculation of Trend & Trend Values

Year	*Sales in '000 quintals (Y)*	*Deviations from 2005 (X)*	X^2	*XY*
2001	70	-4	16	-280
2003	90	-2	4	-180
2005	100	0	0	0
2007	130	2	4	260
2009	170	4	16	680
	$\Sigma Y = 560$	0	$\Sigma X^2 = 40$	$\Sigma XY = 480$

The equation of the straight line trend is

$Y_c = a + bX$

Since $\Sigma X = 0$; $a = \frac{\Sigma Y}{N} = \frac{560}{5} = 112$

$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{480}{40} = 12$

The trend line is $Y_c = 112 + 12X$

(ii) For 2012 X = 7, so Y_{2012} = 112 + 12 (7)

Y = 196.

(iii) The annual increase in the expected, sales of the firm is 12 ('000 quintals)

Example 9 :

The sales of a commodity (in '000 of ₹) are given below :

Year	2001	2002	2003	2004	2005	2006	2007
Sales (in'000 of ₹)	82	86	81	86	92	90	99

(i) Using the method of least squares, fit a straight line equation to the data

(ii) What is the average annual change in the sales?

(iii) Obtain the trend values for the years 2001-2007 and show that the sum of difference between the actual and the trend values is equal to zero.

(iv) What are the expected sales for the year 2012 ?

Solution :

Table : Calculation of trend values

Year	*Sales (in '000 of ₹) (Y)*	*Deviations from 2002 (X)*	X^2	*XY*	*Trend values* $Y_c = 88+2.5X$	$Y- Y_c$
2001	82	-3	9	-246	88 + 2.5(-3) = 80.5	1.5
2002	86	-2	4	-172	88 + 2.5(-2) = 83.0	3.0
2003	81	-1	1	-81	88 + 2.5(-l) = 85.5	-4.5
2004	86	0	0	0	88 + 2.5(0) = 88.0	-2.0
2005	92	1	1	92	88 + 2.5(1) = 90.5	1.5
2006	90	2	4	180	88 + 2.5(2) = 93.0	-3.0
2007	99	3	9	297	88 + 2.5(3) = 95.5	3.5
	$\Sigma Y=616$	$\Sigma X=0$	$\Sigma X^2=28$	$\Sigma XY=70$		$\Sigma Y-Y_c=0$

(i) The equation of the straight line trend is

$Y_c = a + bX$

Since $\Sigma X = 0$; $a = \frac{\Sigma XY}{\Sigma X^2} = \frac{70}{28} = 2.5$

The trend line is $Y_c = 88 + 2.5X$

(i) The average annual change in sales is 2.5 x 1000 = ₹ 2,500

(iii) Sum of difference between the actual- (Y) and trend line (Y_c) is equal to

$\Sigma(Y - Y_c) = 0$ as shown in last column of the table.

(iv) Expected sales for the year 2012 —

For 2012 X = 8, so Y_{2012} = 88 + 2.5(8) = 88 + 20

Y = ₹ 108

The expected sales for the year 2012 is ₹ 1,08,000.

Example 10 :

Fit a straight line trend by the method of least squares taking year 2008 as origin, and estimate the sales for the year 2012 :

Year	2006	2007	2008	2009	2010	2011
Sales (crores)	24	26	28	30	44	52

Solution :

Table : Straight line trend by Method of Least Squares

Year	*Sales (crores)*	*Deviation from 2002 (X)*	X^2	*XY*
2006	24	-2	4	-48
2007	26	-1	1	-26
2008	28	0	0	0
2009	30	1	1	30
2010	44	2	4	88
2011	52	3	9	156
	$\Sigma Y=204$	$\Sigma X = 3$	$\Sigma X^2 = 19$	$\Sigma XY = 200$

equation of a straight line trend is

$Y_c = a + bX$

To get the value of a and b, the following two normal equations have to be solved simultaneously—

$\Sigma Y = Na + b\Sigma X$

$\Sigma XY = a\Sigma X + b\Sigma X^2$

$204 = 6a + 3b$ ......(1)

$200 = 3a + 196$ ......(2)

Multiplying equation (2) by 2 and subtracting from equation (2), we get

$-196 = -35b$

$$b = \frac{-196}{-35} = 5.6$$

Putting the value of *b* in equation 1

$204 = 6a + 3 \text{ X } (5.6)$

$204 = 6a + 16.8$

$6a = 204-16.8$

$$a = \frac{187.2}{6} = 31.2$$

Equation for straight line trend is

$Y_c = 31.2 + 5.6$ X (origin 2008)

For 2012, X = 4

Y_c = 31.2 + 5.6 X 4 = 31.2 + 22.4 = 53.6

Thus, the estimated, sales for the year 2012 is 53.6 crores.

Example 11 :

Fit a straight line trend to the following data by least squares method :

2001	2003	2005	2007	2009
18	21	23	27	16

(i) Estimate sales for the year 2012.

(ii) What is the annual increase/decrease in the trend values of sales ?

Solution :

Table : Least Squares Method

	Exports (in tons)	*Deviation from 2005 X*	X^2	*XY*
2001	18	-4	16	-72
2003	21	-2	4	-42
2005	23	0	0	0
2007	27	2	4	54
2009	16	4	16	64
	$\Sigma Y = 105$	$\Sigma X = 0$	$\Sigma X^2 = 40$	$\Sigma XY = 4$

(i) The equation of the straight line trend is

$Y_c = a + bX$

Since $\Sigma X = 0$; $a = \frac{\Sigma Y}{N} = \frac{105}{5} = 21$

$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{4}{40} = 0.1$

The trend line is $Y_c = 21 + 0.1X$ (origin 2005)

For 2012 X = 7, so $Y_{2012} = 21 + 0.1(7) = 21 + 0.7$

Y = 21.7

The expected sales for the year 2012 is ₹ 21.7 lakh

(ii) The annual increase in the trend values of sales (as given by *b)* is ₹ 0.1 lakh *i.e.* ₹ 10,000.

SELF EXAMINATION QUESTIONS

Problem 1: What are the different components of a time series? Describe briefly each of these components ?

Problem 2: Briefly describe various components of time series. Give the additive & multiplicative models of time series.

Problem 3: What is 'secular trend ? What is the use of studying it ? List two methods of measuring trend.

Study Note - 9

PROBABILITY

This Study Note includes

9.1 GENERAL CONCEPT

The concept of probability is difficult to define in precise terms. In ordinary language, the word probable means likely or chance. The probability theory is an important branch of mathematics. Generally the word, probability, is used to denote the happening of a certain event, and the likelihood of the.occurrence of that event, based on past experiences. By looking at the clear sky, one will say that there will not be any rain today. On the other hand, by looking at the cloudy sky or overcast sky, one will say that there will be rain today. In the earlier sentence, we aim that there will not be rain and in the latter we expect rain. On the other hand a mathematician says that the probability of rain is 0 in the first case and that the probability of rain is 1 in the second case. In between 0 and 1, there are fractions denoting the chance of the event occurring.

If a coin is tossed, the coin falls dawn. The coin has two sides ; head and tail. On tossing a coin, the coin may fall down either with the head up or tail up. A coin, on reaching the ground, will not stand on its edge or rather, we assume ; so the probability of the coin coming down is 1. The probability of the head coming up is 50% and the tail coming up is 50% ; in other words we can say the probability of the head or the tail coming up is $\frac{1}{2}, \frac{1}{2}$ ince 'head' and 'tail' share equal chances. The probability that it will come down head or tail is unity.

9.2 SOME USEFUL TERMS

Before discussing the theory of probability, let us have an understanding of the following terms :

9.2.1. Random Experiment or Trial :

If an experiment or trial can be repeated under the same conditions, any number of times and it is possible to count the total number of outcomes, but individual result i.e. individual outcome is not predictable. Suppose we toss a coin. It is not possible to predict exactly the outcomes. The outcome may be either head up or tail up. Thus an action or an operation which can produce any result or outcome is called a random experiment or a trial.

9.2.2. Event :

Any possible outcome of a random experiment is called an event. Performing an experiment is called trial and outcomes are termed as events.

An event whose occurrence is inevitable when a certain random experiment is performed, is called a sure event or certain event. At the same time, an event which can never occur when a certain random experiment is performed is called an impossible event. The events may be simple or composite. An event is called simple if it corresponds to a single possible outcome. For example, in rolling a die, the chance of getting 2 is a simple event. Further in tossing a die, chance of getting event numbers (1, 3, 5) are compound event.

9.2.3. Sample space

The set or aggregate of all possible outcomes is known as sample space. For example, when we roll a die, the possible outcomes are 1, 2, 3, 4, 5, and 6 ; one and only one face come upwards. Thus, all the outcomes— 1, 2, 3, 4, 5 and 6 are sample space. And each possible outcome or element in a sample space called sample point.

9.2.4. Mutually exclusive events or cases :

Two events are said to be mutually exclusive if the occurrence of one of them excludes the possibility of the occurrence of the other in a single observation. The occurrence of one event prevents the occurrence of the other event. As such, mutually exclusive events are those events, the occurrence of which prevents the possibility of the other to occur. All simple events are mutually exclusive. Thus, if a coin is tossed, either the head can be up or tail can be up; but both cannot be up at the same time.

Similarly, in one throw of a die, an even and odd number cannot come up at the same time. Thus two or more events are considered mutually exclusive if the events cannot occur together.

9.2.5. Equally likely events :

The outcomes are said to be equally likely when one does not occur more often than the others.

That is, two or more events are said to be equally likely if the chance of their happening is equal. Thus, in a throw of a die the coming up of 1, 2, 3, 4, 5 and 6 is equally likely. For example, head and tail are equally likely events in tossing an unbiased coin.

9.2.6. Exhaustive events

The total number of possible outcomes of a random experiment is called exhaustive events. The group of events is exhaustive, as there is no other possible outcome. Thus tossing a coin, the possible outcome are head or tail ; exhaustive events are two. Similarly throwing a die, the outcomes are 1, 2, 3, 4, 5 and 6. In case of two coins, the possible number of outcomes are 4 *i.e.* (2^2), *i.e., HH, HT TH* and TT. In case of 3 coins, the possible outcomes are 2^3=8 and so on. Thus, in a throw of *n*" coin, the exhaustive number of case is 2^n.

9.2.7. Independent Events

A set of events is said to be independent, if the occurrence of any one of them does not, in any way, affect the Occurrence of any other in the set. For instance, when we toss a coin twice, the result of the second toss will in no way be affected by the result of the first toss.

9.2.8. Dependent Events

Two events are said to be dependent, if the occurrence or non-occurrence of one event in any trial affects the probability of the other subsequent trials. If the occurrence of one event affects the happening of the other events, then they are said to be dependent events. For example, the probability of drawing a king from a pack of 52 cards is 4/52, ; the card is not put back ; then the probability of drawing a king again is 3/51. Thus the outcome of the first event affects the outcome of the second event and they are dependent. But if the card is put back, then the probability of drawing a king is 4/52 and is an independent event.

9.2.9. Simple and Compound Events

When a single event take place, the probability of its happening or not happening is known as simple event.

When two or more events take place simultaneously, their occurrence is known as compound event (compound probability) ; for instance, throwing a die.

9.2.10. Complementary Events :

The complement of an events, means non-occurrence *of A* and is denoted by $\overline{A}$. $\overline{A}$ contains those points of the sample space which do not belong to *A*. For instance let there be two events *A* and *B*. *A* is called the complementary event of *B* and vice verse, if *A* and *B* are mutually exclusive and exhaustive.

9.2.11. Favourable Cases

The number of outcomes which result in the happening of a desired event are called favourable cases to the event. For example, in drawing a card from a pack of cards, the cases favourable to "getting a diamond" are 13 and to "getting an ace of spade" is only one. Take another example, in a single throw of a dice the number of favourable cases of getting an odd number are three -1,3 and 5.

9.3 MEASUREMENT OF PROBABILITY

The origin and development of the theory of probability dates back to the seventeenth century. Ordinarily speaking the probability of an event denotes the likelihood of its happening. A value of the probability is a number ranges between 0 and 1. Different schools of thought have defined the term probability differently. The various schools of thought which have defined probability are discussed briefly.

9.3.1. Classical Approach (Priori Probability)

The classical approach is the oldest method of measuring probabilities and has its origin in gambling games. According to this approach, the probability is the ratio of favourable events to the total number of equally likely events. If we toss a coin we are certain that the head or tail will come up. The probability of the coin coming down is 1, of the head coming up is $\frac{1}{2}$ and of the tail coming up is $\frac{1}{2}$. It is customary to describe the probability of one event as 7" (success) and of the other event as '*q*' (failure) as there is no third event.

$$P = -\frac{\text{Number of favourable cases}}{\text{Total number of equally likely cases}}$$

If an event can occur in 'a' ways and fail to occur in '*b*' ways and these are equally to occur, then the probability of the event occurring, $\frac{a}{a+b}$ is denoted by *P*. Such probabilities are also known as unitary or theoretical or mathematical probability. *P* is the probability of the event happening and *q* is the probability of its not happening.

$$P = \frac{a}{a+b} \text{ and } q = \frac{b}{a+b}$$

$$\text{Hence } P + q = \frac{a}{(a+b)} + \frac{b}{(a+b)} = \frac{a+b}{a+b} = 1$$

Therefore

$P + q = 1.\ 1 - p = q,\ 1 - q = p$

Probabilities can be expressed either as ratio, fraction or percentage, such as - or 0.5 or 50%

9.3.1.1. Limitations of Classical Approach:

1. This definition is confined to the problems of games of chance only and cannot explain the problem other than the games of chance.
2. We cannot apply this method, when the total number of cases cannot be calculated.
3. When the outcomes of a random experiment are not equally likely, this method cannot be applied.
4. It is difficult to subdivide the possible outcome of experiment into mutually exclusive, exhaustive and equally likely in most cases.

Example 1:

What is the chance of getting a king in a draw from a pack of 52 cards?

Solution :

The total number of cases that can happen

= 52 (52 cards are there).

Total number of kings are 4 ; hence favourable cases=4 Therefore probability of drawing a king $=\frac{4}{52}=\frac{1}{13}$.

Example 2:

Two coins are tossed simultaneously. What is the probability of getting a head and a tail ?

Solution :

The possible combinations of the two coins turning up with head (*H*) or tail *(T)* are *HH, HT, TH, TT*. The favourable ways are two out of these four possible ways and all these are equally likely to happen.

Hence the probability of getting a head and a tail is $\frac{2}{4}=\frac{1}{2}$.

Example 3 :

One card is drawn at random from a well-shuffled pack of 52 cards. What is the probability that it will be *(a)* a diamond *(b)* a queen ?

Solution :

(a) There are 13 diamond cards in a pack of 52 cards. The number of ways in which a card can be drawn from that pack is 52. The number favourable to the event happening is 13.

Hence probability of drawing a diamond

$$\frac{13}{52}=\frac{1}{4}$$

(b) There are 4 queens in the pack ; and so the number of ways favourable to the event = 4

The probability = $\frac{4}{52}=\frac{1}{13}$

Example 4 :

Two cards are drawn from a pack of cards at random. What is the probability that it will be (a) a diamond and a heart *(b)* a king and a queen (c) two kings ?

Solution :

(a) The number of ways of drawing 2 cards from out of 52 cards

$$= {}^{52}C_2 = \frac{52\times 51}{1\times 2} = 26\times 51$$

The number of ways of drawing a diamond and a heart

=13 x 13

The required probability

$$=\frac{13\times 13}{26\times 51}=\frac{13}{102}$$

(b) The number of ways of drawing a king and a queen = 4 × 4

The required probability

$$= \frac{4 \times 4}{26 \times 51} = \frac{6}{663}$$

(c) Two kings can be drawn out of 4 kings in $^4C_2 = \frac{\lfloor 4}{\lfloor 4-2 . \lfloor 2} = \frac{\lfloor 4}{\lfloor 2 . \lfloor 2} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 6$ ways

The probability of drawing 2 kings

$$= \frac{6}{26 \times 51} = \frac{1}{221}$$

Example 5 :

A bag contains 7 red, 12 white and 4 green balls. What is the probability that :

(a) 3 balls drawn are all white and

(b) 3 balls drawn are one of each colour ?

Solution :

(a) Total number of balls

= 7 +12 + 4 = 23

Number of possible ways of drawing 3 out of 12 white

$= {}^{12}C_3$

Total number of possible ways of drawing 3 out of 23 balls

$= {}^{23}C_3$

Therefore, **probability** of drawing 3 white balls $= \frac{^{12}C_3}{^{23}C_3} = \frac{220}{1771} = 0.1242$

(b) Number of possible ways of drawing 1 out of 7 red — $^7C_1$ Number of possible ways of drawing 1 out of 12 white = $^{12}C_1$ Number of possible ways of drawing 1 out of 4 green - $^4C_1$

Therefore the probability of drawing balls of different colours

$$= \frac{^7C_1 \times {}^{12}C_1 \times {}^4C_1}{^{23}C_3} = \frac{7 \times 12 \times 4}{1771}$$

$= 0.1897$

9.3.2. Relative Frequency Theory of probability :

Classical approach is useful for solving problems involving game of chances—throwing dice, coins, etc. but if applied to other types of problems it does not provide answers. For instance, if a man jumps from a height of 300 feet, the probability of his survival will, not be 50%, since survival and death are not equally alike.

Similarly, the prices of shares of a Joint Stock Company have three alternatives i.e. the prices may remain constant or prices may go up or prices may go down. Thus, the classical approach fails to answe questions of these type.

If we toss a coin 20 times, the classical probability suggests that we; should have heads ten times. But in practice it may not be so. These empirical approach suggests, that if a coin is tossed a large number of time, say, 1,000 times, we can expect 50% heads and 50% tails. Vor Miscs explained, "If the experiment be repeated a large number of times under essentially identical conditions, the limiting value of the ratio of the number of times the event *A* happens to the total, number of trials.of the experiments as the number of trials increases indefinitely, is called the probability of the occurrence of *A*".

Thus, $P(A) = \lim_{n\to\infty} \frac{m}{n}$

The happening of an event is determined on the basis of past experience or on the basis of relative frequency of success in the past. For instance, a machine produces 10% unacceptable articles of the total output. On the basis of such experience or experiments, we may arrive at that (i) the relative frequency obtained on the basis of past experience can be shown to come very close to the classical probability. For example, as said earlier, a coin is tossed for 6 times, we may not get exactly 3 heads and 3 tails. But, the coin is tossed for larger number of times, say 10,000 times, we can expect heads and tails very close to 50% (ii) There are certain laws, according to which the 'occurrence' or 'non-occurrence of the events take place. Posterior probabilities, also called Empirical Probabilities are based on experiences of the past and on experiments conducted. Thus, relative frequency can be termed as a measure of probability and it is calculated on the basis of empirical or statistical findings. For instance if a machine produces 100 articles in the past, 2 particles were found to be defective, then the probability of the defective articles is 2/100 or 2%.

9.3.2.1. Limitations of Relative Frequency Theory of Probability:

1. The experimental conditions may not remain essentially homogeneous and identical in a large number of repetitions of the experiment.
2. The relative frequency —, may not attain a unique value no matter however large N may be.
3. Probability **P(A) defined** can never be obtained in practice. We can **only attempt** at a close estimate of P(A) by making N sufficiently large.

Example 6:

An urn contains 8 white and 3 red balls. If two balls are drawn at random, find the probability that *(a)* both are white, *(b)* both are red and (c) one is of each colour.

Solution :

Total number of balls in the urn = 8 + 3 =11

Two balls can be drawn out of 11 balls in ${}^{11}C_2$ ways.

Exhaustive number of cases = ${}^{11}C_2 = \frac{11\times10}{2} = 55$.

(a) Two white balls to be drawn out of 8 white, can be done in ${}^{8}C_2 = \frac{8\times7}{2} = 28$ ways.

The probability that both are white = $= \frac{28}{55}$

(b) Two red balls to be drawn out of 3 red balls can be done in ${}^{3}C_2$=3 ways.

Hence, the probability that both are red $= \frac{3}{55}$

(c) The number of favourable cases for drawing one white ball and one red ball is

${}^{8}C_1 x {}^{3}C_1$= 8 x 3 = 24.

Therefore, the probability (one red and one white) $=\frac{24}{55}$

Example 7 :

Tickets are numbered from 1 to 100. They are well shuffled and a ticket is drawn at random. What is the probability that the drawn ticket has :

(a) an even number,

(b) a number 5 or a multiple of 5,

(c) a number which is greater than 75,

(d) a number which is a square ?

Solution :

(a) The total number of exhaustive, mutually exclusive and equal cases is 100. There are 50 even numbered tickets.

Therefore, favourable cases to the event is 50.

Therefore, the probability $=\frac{50}{100}=\frac{1}{2}$

(b) Suppose A denotes the number of happenings that the drawn ticket has a number 5 or a multiple of 5. These are 20 cases i. e., 5, 10, 15, 20,...100.

Therefore, $P(A)=\frac{20}{100}=\frac{1}{5}$

(c) There are 25 cases, which have a number greater than 75. Say A will denote it.

Therefore, $P(A)=\frac{25}{100}=\frac{1}{4}$

(d) There are 10 favourable cases which give squares between 1 and 100 i.e., 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Therefore, $P(A)=\frac{10}{100}=\frac{1}{10}$

Example 8 :

Four cards are drawn from a pack of 52 cards without replacement. What is the probability that they are all of different suits ?

Solution :

The required probability would be :

$$1\times\frac{39}{51}\times\frac{26}{50}\times\frac{13}{49}=\frac{2,197}{20,825}$$

9.4 THEOREMS OF PROBABILITY

We have studied what probability is and how it can be measured. We dealt with simple problems. Now we shall consider some of the laws of probability to tackle complex situation. There are two important theorems, viz., (1) the Addition Theorem and (2) the Multiplication Theorem.

9.4.1. Addition Theorem :

The simplest and most important **rule used in** the **calculation is** the **addition rules, it** states, "If two events are mutually **exclusive, then** the probability of the occurrence of either *A* or *B* is the **sum** of **the** probabilities of *A* and *B*. Thus,

$P(A \text{ or } B)=P(A)+P(B)$

Example 9 :

A bag **contains** 4 white, 3 black and 5 red balls. What is the probability of getting a white or a red ball at random in a single draw ?

Solution :

The probability of getting a white ball = $\frac{4}{12}$

The probability of getting a red ball = $\frac{5}{12}$

The probability of a white or a red $=\frac{4}{12}+\frac{5}{12}=\frac{9}{12}$

or $\frac{9}{12}\times 100=75\%$

When events are not mutually exclusive

The addition theorem studied above is not applicable when the events are not mutually exclusive. In such cases where the events are not mutually exclusive, the probability is :

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Example 10 :

Two students X and Y work independently on a problem. The probability that A will solve it is 3/4 and the probability that *Y* will solve it is 2/3. What is the probability that the problem will be solved ?

Solution :

$$P(A \text{ or } B)=P(A)+P(B)-P(A \text{ and } B)$$

The probability that X will solve the problem is = 3/4

The probability that *Y* will solve the problem is—2/3

The events are not mutually exclusive as both of them may solve the problem.

Therefore, the probability = $=\frac{3}{4}+\frac{2}{3}-\frac{3}{4}\times\frac{2}{3}$

$=\frac{17}{12}-\frac{6}{12}=\frac{11}{12}$

Alternatively:

The probability that *X* will solve it and *Y* fail to solve it = 3/4 x 1/3 = 3/12

$\therefore$ Probability that the problem will be solved $=\frac{2}{3}+\frac{3}{12}=\frac{11}{12}$

Alternatively

The probability that *X* will fail to solve and will *Y* solve it

= 1/4 x 2/3 = 2/12

$\therefore$ Probability that the problem will be solved $=\frac{3}{4}+\frac{2}{12}=\frac{9+2}{12}=\frac{11}{12}$

Alternatively :

The probability that neither *X* nor *Y* will solve it $\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$

Hence, the probability that the problem will be solved

$=1-\frac{1}{12}=\frac{11}{12}$

9.4.2. Multiplication

When it is desired to estimate the chances of the happening of successive events, the separate probabilities of these successive events are multiplied. If two events *A* and *B* are independent, then the probability that both will occur is equal to the product of the respective probabilities. We find the probability of the happening of two or more events in succession.

Symbolically :

$$P(A \text{ and } B)=P(A) x P(B)$$

Example 11 :

In two tosses of a fair coin, what are the chances of head in both ?

Solution :

Probability of head in first toss = 1/2

Probability of head in the second toss = 1/2

Probability of head in both tosses = 1/2x1/2 -1/4

Example 12 :

The probability that *X* and *Y* will be alive ten years hence is 0.5 and 0.8 respectively. What is the probability that both of them will be alive ten years hence ?

Solution :

Probability of *X* being alive ten years hence = 0.5

Probability of *Y* being alive ten years hence = 0.8

Probability of *X* and *Y* both being alive ten years hence =.5x.8=0.4

When events are dependent :

If the events are dependent, the probability is conditional. Two events *A* and *B* are dependent ; *B* occurs only when *A* is known to have occurred.

P (B | A) means the probability of B given that A has occurred.

$$P(B \mid A) = \frac{P(AB)}{P(A)}; \; P\left(\frac{A}{B}\right) = \frac{P(AB)}{P(B)}$$

Example 13 :

A man want to marry a girl having qualities: White complexion the probability of getting such girl is 1 in 20. Handsome dowry - the probabihty of getting is 1 in 50. Westernised style - the probability is 1 in 100.

Find out the probability of his getting married to such a girl, who has all the three qualities.

Solution :

The probability of a girl with white complexion $= \frac{1}{20}$ or 0.05. The probability of a girl with handsome dowry $= \frac{1}{50}$ or 0.02. The probabihty of a girl with westernised style $= \frac{1}{100}$ or 0.01. Since the events are independent, the probabihty of simultaneous occurrence of all three qualities =

$$\frac{1}{20} \times \frac{1}{50} \times \frac{1}{100} = 0.05 \times 0.02 \times 0.01 = 0.00001$$

Example 14 :

A university has to select an examiner from a list of 50 persons, 20 of them women and 30 men, 10 of them knowing Hindi and 40 not. 15 of them being teachers and the remaining 35 not. What is the probability of the University selecting a Hindi-knowing women teacher ?

Solution :

Probability of selecting a women $= \frac{20}{50}$

Probability of selecting a teacher $= \frac{15}{50}$

Probability of selecting a Hindi-knowing candidate $= \frac{10}{50}$

Since the events are independent the probabihty of the University selecting a Hindi-knowing woman teacher is :

$$\frac{20}{50} \times \frac{15}{50} \times \frac{10}{50} = \frac{3}{125} \text{ or } 0.024.$$

Example 15 :

A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. Determine the probability that it is :

(i) Red (ii) white, (iii) Blue, (iv) Not Red and (v) Red or White.

Solution :

$$P = \frac{\text{No. of favourable cases}}{\text{Total No. of equally likely cases}}$$

(i) Probability of Red $= \frac{6}{15}$ or 0.40

(ii) Probability of white $= \frac{4}{15}$ or 0.267

(iii) Probability of Blue $= \frac{5}{15}$ A or 0.333

(iv) Probability of not Red $= \frac{9}{15}$ or 0.60

(v) Probability of Red and White $= \frac{10}{15}$ or 0.667

9.5 BAYES' THEOREM

This theorem is associated with the name of Reverend **Thomas Bayes**. It is also known as the inverse probability. Probabilities can be revised when new information pertaining to a random experiment is obtained. One of the important applications of the conditional probability is in the computation of unknown probabilities, on.the basis of the information supplied by the experiment or past records. That is, the applications of the results of probability theory involves estimating unknown probabilities and making decisions on the basis of **new** sample information. This concept is referred to as Bayes' Theorem. Quite often the businessman has the extra information on a particular event, either through a personal belief or from the past history of the events. Revision of probability arises from a need to make better use of experimental information. Probabilities assigned on the basis of personal experience, before observing the outcomes of the experiment are called prior probabilities. For example, probabilities assigned to past sales records, to past number of defectives produced by a machine, are examples of prior probabilities. When the probabilities are revised with the use of Bayes' rule, they are called posterior probabilities. Bayes' theorem is useful in solving practical business problems in the light of additional information. Thus popularity of the theorem has been mainly because of its usefulness in revising a set of old probability (Prior Probability) in the light of additional information made available and to derive a set of new probabilily (i.e. Posterior Probability)

Bayes' Theorem : An event A can occurre only if one of the mutually exclusive and exhaustive set of events B_1, B_2, ... B_n occurs. Suppse that the unconditional probabilities

$$P(B_1), P(B_2), \ldots P(B_n)$$

and the conditional probabilities

$$P(A/B_1), P(A/B_2), \ldots P(A/B_n)$$

are known. Then the conditional probability $P(B_i/A)$ of a specific event B_i, when A is stated to have actually accquared, is given by

$$P(B_i / A) = \frac{P(B_i).P(A / B_i)}{\sum_{i=i}^{n} P(B_i).P(A / B_i)}$$

This is known as Bayes' Theorem.

The following example illustrate the application of Baye's Theorem.

The above calculation can be verified as follows :

If 1,000 scooters were produced by the two plants in a particular week, the number of scooters produced by Plant I & Plant II are respectively :

l,000 x 80% = 800 scooters

1,000 x 20% = 200 scooters

The number of standard quality scooters produced by Plant I :

800 x 85/100 = 680 scooters

The number of standard quality scooters produced by Plant II :

200 x 65/100 = 130 Scooters.

The probability that a standard quality scooter was produced by Plant I is :

$$=\frac{680}{680+130}=\frac{680}{810}=\frac{68}{81}$$

The probability that a standard quality scooter was produced by Plant II is :

$$=\frac{130}{680+130}=\frac{130}{810}=\frac{13}{81}$$

The same process *i.e.* revision can be repeated if more information is made available. Thus it is a good theorem in improving the quality of probability in decision making under uncertainty.

Example 16 :

You note that your officer is happy on 60% of your calls, so you assign a probability of his being happy on your visit as 0.6 or 6/10. You have noticed also that if he is happy, he accedes to your request with a probability of 0.4 or 4/10 whereas if he is not happy, he acedes to the request with a probability of 0.1 or D or $\frac{1}{10}$. You call one day, and he accedes to your request. What is the probability of his being happy ?

Solution :

Let- *H* be the Hypothesis that the officer is happy and $\overline{H}$ the Hypothesis that the officer is not happy

$$P(H)=\frac{6}{10} \qquad P(\overline{H})=\frac{4}{10}$$

Let *A* be the event that he accedes to request

$$P(A/H)=\frac{4}{10}, \quad P(A/\overline{H})=\frac{1}{10}$$

To find P(H/A), according to Baye's Theorem,

$$P(H/A)=\frac{P(H)\times P(A/H)}{P(H)\times P(A/H)+P(\overline{H})\times P\ \frac{A}{\overline{H}}}=\frac{\frac{6}{10}\times\frac{4}{10}}{\frac{6}{10}.\frac{4}{10}+\frac{4}{10}.\frac{1}{10}}$$

$$= \frac{\frac{24}{100}}{\frac{24}{100}+\frac{4}{100}} = \frac{24}{28} = \frac{6}{7} = 0.857$$

Example 17 :

A company has two plants to manufacture scooters. Plant I manufactures 80% of the scooters and plant II manufactures 20%. At Plant I, 85 out of 100 scooters are rated standard quality or better. At Plant II, only 65 out of 100 scooters are rated standard quality or better. What is the probability that the scooter selected at random came from Plant I if it is known that the scooter is of standard quality ?

What is the probability that the scooter came from Plant II if it is known that the scooter is of standard quality.

Solution :

Let A_1 be the event of drawing a scooter produced by Plant I and A_2 be the event of drawing a scooter produced by Plant II. *B* be the event of drawing a standard quality scooter produced by either Plant I or Plant II

Then, from the first information :

$$P(A_1) = \frac{80}{100} = 80\% = 0.80$$

$$P(A_2) = \frac{20}{100} = 20\% = 0.2$$

From the additional information :

$$P(B \mid A_1) = \frac{85}{100} = 85\%; \quad P(B \mid A_2) = 65\%$$

The required values are computed in the following table :

Event	*Prior Probability(2)*	*Conditional Probability(3)*	*Joint Probability (4)*	*Posterior Probability (Revised)(5)* [4 ÷ P(B)]
A_1	0.80	0.85	0.68	$\frac{0.68}{0.81} = \frac{68}{81}$
A_2	0.20	0.65	0.13	$\frac{0.13}{0.81} = \frac{13}{81}$
	1		P(B) = 0.81	1

From the first information we may say that the standard scooter is drawn from Plant I since $P(A_1) = 80\%$ which is greater than $P(A_2) = 20\%$,

From the additional information i.e. at Plant I, 85 out of 100 and at Plant II 65 out of 100 are rated standard quality, we can give better answer, Thus we may conclude that the standard quality of scooter is more likely drawn from the output by Plant I.

Example 18 :

Box I contains three defective and seven non-defective balls, and Box II contains one defective and nine non-defective balls. We select a box at random and then draw one ball at random from the box.

(a) What is the probability of drawing a non-defective ball ?

(b) What is the probability of drawing a defective ball ?

(c) What is the probability that box I was chosen, given a defective ball is drawn ?

Solution :

$P(B_1)$ or Probability that Box I is chosen $= \frac{1}{2} P(B_1)$ or

Probability that Box I is chosen $= \frac{1}{2}$.

$P(B_2)$ or Probability that Box II is chosen $= \frac{1}{2}$

$P(D)$ - Probability that a defective Ball is drawn $P(ND)$ = Probability that a non-defective Ball is drawn

Joint Probability

$\frac{1}{2} \times \frac{3}{10} = \frac{3}{20}$ $\frac{1}{2} \times \frac{1}{10} = \frac{1}{20}$

$\frac{1}{2} \times \frac{7}{10} = \frac{7}{20}$ $\frac{1}{2} \times \frac{9}{10} = \frac{9}{20}$

(a) $P(ND)=P$ (Box I and non-defective) +P(Box II non-defective)

$$= \frac{1}{2} \times \frac{7}{10} + \frac{1}{2} \times \frac{9}{10} = \frac{16}{20}$$

(b) $P(D)$ = P (Box I and defective) + P (Box II and defective)

$$= \frac{1}{2} \times \frac{3}{10} + \frac{1}{2} \times \frac{1}{10} = \frac{4}{20}$$

(c) Bayes' Theorem :

$$P(B_1 / D) = \frac{P(B_1 \text{ and} D)}{P(D)} = \frac{3/20}{4/20} = \frac{3}{4}$$

$P(B_1)$ and $P(B_2)$ are called prior probabilities and $P(B_1/D)$ and $P(B_2/D)$ are called posterior probabilities. The above information is summarised in the following table :

Event	*Prior Probability*	*Conditional Probability*	*Joint Probability*	*Posterior Probability*
B_1	$\frac{1}{2}$	*3/10*	*3/20*	*3/4*
B_2	$\frac{1}{2}$	*1/10*	*1/20*	*1/4*
	1		*4/10*	*1*

9.6 ODDS

We must know the concept of odds. The word odd is frequently used in statistics. Odds relate the chances in favour of an event to the chances against it. For instance, the odds are 2 : 1 that *A* will get a job, means that there are 2 chances that he will get the job and 1 chance against his getting the job. This can also be converted into probability as getting the job = 2/3. Therefore, if the odds are *a* : *b* in favour of an event then *P(A) = a/(a+b)*. Further, it may be noted that the odds are *a : b* in favour of an event is the same as to say that the odds are *b : a* against the event.

If the probability of an event is *p,* then the odds in favour of its occurrence are P to (*1-p*) and the odds against its occurrence are 1-*p to p*.

Example 19 :

Suppose it is 11 to 5 against a person who is now 38 years of age living till he is 73 and 5 to 3 against *B* who is 43 Living till he is 78, find the chance that at least one of these persons will be alive 35 years hence.

Solution :

The probability that *A* will die within 35 years $=\frac{11}{16}$

The probability that *B* will die within 35 years $=\frac{5}{8}$

The probability that both of them will die within 35 years

$$=\frac{11}{16}\times\frac{5}{8}=\frac{55}{128}$$

The probability that both of them will not die i.e.

atleast one of them will be alive

$$=1-\frac{55}{128}=\frac{73}{128}$$

or $\frac{73}{128}\times100=57\%$

Example 20 :

Two cards are drawn at random from a well-shuffled pack of 52 cards. What is the probability that :

(a) both are aces,

(b) both are red,

(c) at least one is an ace ?

Solution :

(a) Let *A* indicate the event of drawing 2 aces.

$$P\ \frac{A}{A} = P(A)\times P\ \frac{A}{A}\ \ s$$

P(A) : drawing of an ace first

$P\ \frac{A}{A}$: conditional probability of an ace at the second draw, given that the first was an ace.

Therefore,

$$P\ \frac{A}{A} = \frac{4}{52} P\ \frac{A}{A} = \frac{3}{51}$$

$$P\ \frac{A}{A} = \frac{4}{52}\times\frac{3}{51} = \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$$

(b) Let R indicate the event of drawing 2 red cards

$$P\ \frac{R}{R} = P(R)\times P\ \frac{R}{R}$$

$$= \frac{26}{52}\times\frac{25}{51} = \frac{650}{2652} = \frac{25}{102}$$

(c) Let *E* indicate the event of drawing an ace. Then the probability that at least an ace is drawn is denoted by *P(E)*. Probability of not drawing an ace :

$$P\ \frac{E}{E} = P(E)\times P\ \frac{E}{E}$$

$$= \frac{48}{52}\times\frac{47}{51} = \frac{2256}{2652} = \frac{188}{221}$$

Therefor, probability of drawing at least one ace

$$= 1 - \frac{188}{221} = \frac{33}{221}$$

Example 21 :

The odds in favour of a certain event are 2 to 5 and the odds against another event independent of the former are 5 to 6. Find the chance that one at least of the events will happen.

Solution :

The chance that the 1st event happens and the 2nd one does not happen

$$= \frac{2}{7}\times\frac{5}{11} = \frac{10}{77}$$

The chance that the 1st event does not happen and the 2nd happens.

$$= \frac{5}{7}\times\frac{6}{11} = \frac{30}{77}$$

The chance that both the events happen

$$=\frac{2}{7}\times\frac{6}{11}=\frac{12}{77}$$

The chance that one at least of the events will happen,

$$=\frac{10}{77}+\frac{30}{77}+\frac{12}{77}=\frac{52}{77}$$

Alternatively :

The first event does not happen

$$=1-\frac{2}{7}=\frac{5}{7}$$

The second event does **not** happen

$$=1-\frac{6}{11}=\frac{5}{11}$$

$\therefore$ The chance that both do not happen

$$=\frac{5}{7}\times\frac{5}{11}=\frac{25}{77}$$

$$=1-\frac{25}{77}=\frac{52}{77}$$

The chance that one at least will happen

Example 22 :

What is the chance that a leap year, selected at random will contain 53 Sundays ?

Solution:

As a leap year consist of 366 days it contains 52 complete weeks and two more days.

The two consecutive days make the following combinations :

(a) Monday and Tuesday

(b) Tuesday and Wednesday

(c) Wednesday and Thursday

(d) Thursday and Friday

(e) Friday and Saturday

(f) Saturday and Sunday, and

(g) Sunday and Monday

If (f) or (g) occur, then the year consists of 53 Sundays.

Therefore the number of favourable cases = 2

Total number of cases = 7

The probability = $\frac{2}{7}$

Example 23 :

A problem in statistics is given to three students *A, B, C* whose chances of **solving** it are 1/2, 1/3, 1/4 respectively. What is the probability **that the** problem will be solved ?

Solution:

The probability that A fails to soive the problem $=1-\frac{1}{2}=\frac{1}{2}$

The probability that *B* fails to solve the problem $=1-\frac{1}{3}=\frac{2}{3}$

The probability that C fails to solve the problem $=1-\frac{1}{4}=\frac{3}{4}$

The probability that the problem is not solved by *A, B* and C $=\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=\frac{1}{4}$

Therefore, the probability that the problem is solved $=1-\frac{1}{4}=\frac{3}{4}$

Example 24 :

An ordinary die is tossed twice and the difference between the number of spots turned up is noted. Find the probability of a difference of 3.

Solution :

The sample space consists of 36 values.

The event space has the following 6 cases : (1, 4), (2, 5), (3, 6), (4, 1), (5, 2), (6, 3)

The required probability = $\frac{6}{36}$

Example 25 :

From a pack of 52 cards, two cards are drawn at random ; find the chance that one is a knave and the other a queen.

Solution :

Sample space = $^{52}C_2$

Event space =$^{4}C_1 \times {}^{4}C_1$

(as there are 4 queens and 4 knaves in the pack)

Required Probability = $=\frac{^{4}C_1 \times {}^{4}C_1}{^{52}C_2}=\frac{8}{663}$

Example 26 :

A bag contains 7 red balls and 5 white balls. 4 balls are drawn at random. What is the probability that (i) all of them are red ; (ii) two of them are red and two white ?

Solution :

(i) Favourable cases $^7C_4$, Exhaustive cases $^{12}C_4$

Probability $= \frac{^7C_4}{^{12}C_4} = \frac{105}{495} = \frac{7}{33}$

(ii) Favourable cases = $^7C_2 x\, ^5C_2$

Exhaustive cases = $^{12}C_4$

Probability $= \frac{^7C_2 \times {^5C_2}}{^{12}C_4} = \frac{12 \times 10}{495} = \frac{14}{33}$

Example 27 :

A petrol pump proprietor sells on an average ₹ 80,000 worth of petrol on rainy days and an average of ₹ 95,000 on clear days. Statistics from the Metereological Department show that the probability is 0.76 for clear weather and 0.24 for rainy weather on coming Monday. Find the expected value of petrol sale on coming Monday.

Solution :

X_1 = ₹ 80,000; $P_1 = 0.24$

X_2 = ₹ *95,000* $P_2 = 0.76$

The required probability = $P_1X_1 + P_2X_2$

= 0.24 X 80,000 + 0.76 X 95,000

= 19,200 + 72,200 = ₹ 91,400.

The expected value of petrol sale on coming Monday = ₹ 91,400

Example 28 :

A bag contains 6 white and 9 black balls. Two drawings of 4 balls are made such that (a) the Balls are replaced before the second trial (b) the balls are not replaced before the second trial. Find the probability that the first drawing will give 4 white and the second 4 black balls in each case.

Solution :

(a) When the balls are replaced before the second trial the number of ways in which 4 balls may be drawn is $^{15}C_4$

The number of ways in which 4 white balls may be drawn = $^6C_4$

The number of ways in which 4 black balls may be drawn = $^9C_4$

Therefore, the probability of drawing 4 white balls at first trial

$= \frac{^6C_4}{^{15}C_4} = \frac{1}{91}$

The Second trial of drawing 4 black balls.

$= \frac{^9C_4}{^{15}C_4} = \frac{9 \times 8 \times 7 \times 6}{4!} \times \frac{4!}{15 \times 14 \times 13 \times 12} = \frac{6}{65}$

Therefore the chance of the Compound event = $= \frac{1}{91} \times \frac{6}{65} = \frac{6}{5915}$

(b) When the balls are not replaced :

At the first trial, 4 balls may be drawn in ${}^{15}C_4$ ways and 4 white balls may be drawn in ${}^{6}C_4$ ways.

Therefore the chance of 4 white balls at first trial $= \frac{{}^{6}C_4}{{}^{15}C_4} = \frac{1}{91}$ (as above)

When 4 white balls have been drawn and removed, the bag contains 2 white and 9 black balls.

Therefore at the second trial, 4 balls may be drawn in ${}^{9}C_4$ ways and 4 black balls maybe drawn in ${}^{9}C_4$ ways So, the chance of 4 black balls at the second trial

$$= \frac{{}^{9}C_4}{{}^{11}C_4}$$

$$= \frac{9 \times 8 \times 7 \times 6}{4!} \times \frac{4!}{11 \times 10 \times 9 \times 8} = \frac{21}{55}$$

Therefore the chance of the compound event $= \frac{1}{91} \times \frac{21}{55} = \frac{3}{715}$

Example 29 :

A salesman is known to sell a product in 3 out of 5 attempts while another salesman is 2 out of 5 attempts. Find the probability that (i) No sale will be effected when they both try to sell the product and (ii) Either of them will succeed in selling the product.

Solution :

Let the two salesmen be *A* and *B*.

P (A) = The probability that the salesman A is able to sell the

product = $\frac{3}{5}$

P (B) = The probability that the salesman B is able to sell the product $\frac{2}{5}$

(i) probability that no sale will be effected = $1 - \frac{3}{5} \quad 1 - \frac{2}{5} = \frac{6}{25}$

(ii) probability that either of them will succeed in selling the product

$$= \frac{3}{5} + \frac{2}{5} - \frac{3}{5} \times \frac{2}{5} = \frac{19}{25}$$

Example 30 :

A class consists of 100 students, 25 of them are girls and 75 boys, 20 of them are rich and remaining poor, 40 of them are fair complexioned. What is the probability of selecting a fair complexsioned rich girl ?

Solution:

Probability of selecting a fair complexioned student = $=\frac{40}{100}=\frac{2}{5}$

Probability of selecting a rich student $=\frac{20}{100}=\frac{1}{5}$

Probability of selecting a girl $=\frac{25}{100}=\frac{1}{4}$

Since the events are independent, by multiplication rule of probability, the

probability of selecting a fair complexioned rich girl $=\frac{2}{5}\times\frac{1}{5}\times\frac{1}{4}=\frac{2}{100}=0.02$

Example 31 :

Three groups of workers contain 3 men and one woman, 2 man and 2 women, and 1 man and 3 woman respectively. One worker is selected at random from each group. What is the probability that the group selected consists of 1 man and 2 woman ?

Solution :

There are three possibilities :

(i) Man is selected from the first group and women from second and third groups; or

(ii) Man is selected from the second groups and women from first and third groups; or

(iii) Man is selected from the third groups and women from first and second groups.

$\therefore$ the probability of selecting a group of one man & two woman

$$= \frac{3}{4}\times\frac{2}{4}\times\frac{3}{4} + \frac{2}{4}\times\frac{1}{4}\times\frac{3}{4} + \frac{1}{4}\times\frac{1}{4}\times\frac{2}{4}$$

$$= \frac{18}{64}+\frac{6}{64}+\frac{2}{64}=\frac{13}{32}.$$

Study Note - 10

THEORETICAL DISTRIBUTION

This Study Note includes

10.1 THEORETICAL DISTRIBUTION

Broadly speaking, the frequency distributions are of two types : Observed Frequency Distribution and Theoretical Frequency Distribution. The distributions, which are based on actual data or experimentation are called the observed frequency distribution. On the other hand, the distributions based on expectations on the basis of past experience is known as Theoretical Frequency Distribution or Expected Frequency Distribution or Probability Distributions. In short, the observed frequency distribution is based on actual sample studies whereas the theoretical distribution is based on expectations on the basis of previous experience or theoretical considerations. For example, we toss a coin 200 times. We may get 80 heads and 120 tails ; but our expectation is 100 heads and 100 tails, because the chance is 50% heads and 50% tails. On the basis of this expectation we can test whether a given coin is unbiased or not. If a coin is tossed 100 times we may get 40 heads and 60 tails. This is our observation. Our expectation is 50% heads and 50% tails. Now the question is whether this discrepancy is due to sampling fluctuation or is due to the fact that the coin is biased. The word expected or expectation is used in the sense of an average. When a coin is tossed for a large number of times, we will on an average get close to 50% heads and 50% tails. The following are important distributions.

1. Binomial Distribution	Discrete Probability Distribution
2. Poisson Distribution	Discrete Probability Distribution
3. Normal Distribution	Continuous Probability Distribution.

10.2 BINOMIAL DISTRIBUTION

This distribution was discovered by a Swiss mathematician Jame Bernoulli (1654-1705) and is also known as Bernoulli Distribution. He discovered this theory and published it in the year 1700 dealing with dichotomous classification of events one possessing and the other not possessing. The probability of occurrence of an event is p and its non-occurrence is q. The distribution can be used under the following conditions :

1. The number of trials is finite and fixed.
2. In every triail there are only two possible outcomes success or failure.
3. The trials are independent. The outcome of one trial does not affect the other trial.
4. p, the probability of success from trial to trial is fixed and q the probability of failure is equal to 1-p. This is the same in all the trials.

For instance, a card is drawn from a pack of 52 cards. The probability of getting a king is 4/52. Before a second draw, the card drawn is replaced. But if the card is not replaced, we cannot have binomial distribution.

Another example, a head or a tail can be had on a toss of coin ; a card drawn may be black or red ; an item inspected from a batch may be defective or non-defective. In each experiment the outcome can be classified as success or failure. Success is generally denoted by p and failure is $1 - p = q$.

If a single coin is tossed, the outcomes are two: head or tail Probability of head is $\frac{1}{2}$ and tail is $\frac{1}{2}$. Thus $(q+p)^n = \left(\frac{1}{2}+\frac{1}{2}\right)^1 = 1$.

If two coins are thrown, the outcomes are four :

HH	*TH*	*HT*	*TT*
PP	*qp*	*pq*	*qq*
p^2	$2pq$		q^2

Thus for two coins $(p+q)^2 = p^2+2pq+q^2$. This binomial expansion is called binomial distribution.

Thus when coins A and B are tossed, the outcomes are : A and B fall with heads up, A head up and B tails up, A tail up and B head up and A and B fall with tail up.

Probability of 2 heads = $p \times p = p^2$

Probability of 1 head and 1 tail = $(p\times q) + (q\times p) = (pq+pq) = 2pq$

Probability of 2 tails = $q \times q = q^2$. The sum is $p^2 + 2pq + q^2$ as the expansion of $(p+q)^2$.

If three coins are tossed, the following are the outcomes $(p+q)^3 = p^3+3p^2q+3q^2p+q^3$ (*p* for head and *q* for tail). The outcomes are :

HHH	*HHT*	*HTH*	*THH*	*HTT*	*THT*	*TTH*	*TTT*
p^3	p^2q	p^2q	qp^2	pq^2	pq^2	q^2p	q^3

$= p^3 + 3p^2q + 3q^2p + q^3 = (p+q)^3$

When $p = \dfrac{1}{2}$ and $q = \dfrac{1}{2}$, the probability of outcome

$$\left(\frac{1}{2}+\frac{1}{2}\right)^3 = 1/8 + 3/8 + 3/8 + 1/8 = 1.$$

Thus a simple rule to find out the probabilities of 3H, 2H, 1H, 0H is as follows:

3 heads = $p^3 = \left(\frac{1}{2}\right)^3 = \dfrac{1}{8}$

2 heads = $3p^2q = 3\times\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \dfrac{3}{8}$

2 head = $3pq^2 = 3\times\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \dfrac{3}{8}$

0 head = $q^3 = \left(\frac{1}{2}\right)^3 = \dfrac{1}{8}$

Thus in term of binomial expansion it is

$= (p + q)^n$

10.2.1. OBTAINING BINOMIAL COEFFICIENT

For *n* trials the binomial probability distribution consists of (*n* +1) terms, the successive binomial coefficient being ${}^nC_o$, ${}^nC_1$, ${}^nC_2$, ... ${}^nC_{n-1}$, ${}^nC_n$.

To find the terms of the expansion, we use the expansion of $(p+q)^n$. Since ${}^nC_o = {}^nC_n = 1$, the first and last coefficient will always be one. Binomial coefficient will be symmetric form. The values of the binomial coefficient for different values of *n* can be obtained easily form Pascal's triangle given below :

PASCAL'S TRIANGLE

(Showing coefficients of terms $(p + q)^n$)

Value of n	*Binomial coefficients*	*Sum (2^n)*
1	1 1	2
2	1 2 1	4
3	1 3 3 1	8
4	1 4 6 4 1	16
5	1 5 10 10 5 1	32
6	1 6 15 20 15 6 1	64
7	1 7 21 35 35 21 7 1	128
8	1 8 28 56 70 56 28 8 1	256
9	1 9 36 84 126 126 84 36 9 1	512
10	1 10 45 120 210 252 210 120 45 10 1	1,024

It can be easily seen that taking the first and last terms as 1, each term in the above can be obtained by addingthe two-terms on either side of it in the preceding line i.e. the line above it. For instance, in line four, 6 is obtained by adding 3 and 3 in the third line; in line ten, 120 is obtained by adding 36 and 84 and in the same line 120 is obtained by adding 84 and 36 and so on.

Probability for Number of Heads (Successes)

Number of successes (x)	Probability (p)
0	${}^nC_0 p^0 q^n = q^n$
1	${}^nC_1 pq^{n-1}$
2	${}^nC_2 p^2 q^{n-2}$
3	${}^nC_3 p^3 q^{n-3}$
r	${}^nC_r p^r q^{(x-r)}$
n	${}^nC_n p^n q^0 = p^n$

Example 1 :

A coin is tossed six times. What is the probability of obtaining (a) 4 heads, (b) 5 heads, (c) 6 heads and (d)getting 4 or more heads :

Solution :

(a) Probability of 4 heads

$$= {}^6C_4 p^4 q^2 = 15 \times \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 = 0.234$$

(b) Probability of 5 heads

$$= {}^6C_5 p^5 q = 6 \times \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right) = 0.094$$

(c) Probability of 6 heads

$$= {}^6C_6 p^6 q^0 = 1 \times \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^0 = 0.016$$

(d) Probability of getting 4 or more heads

= 0.234 + 0.094 + 0.016 = 0.344

Alternatively

Probability of getting at least 4 heads (means we may get 4 heads or 5 heads or 6 heads)

$p(x = 4) \quad = P(4) + P(5) + P(6)$

$$= {}^6C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 + {}^6C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right) + {}^6C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^0$$

$$= {}^6C_4 \left(\frac{1}{2}\right)^6 + {}^6C_5 \left(\frac{1}{2}\right)^6 + {}^6C_6 \left(\frac{1}{2}\right)^6$$

$$= \left(\frac{1}{2}\right)^6 \left({}^6C_4 + {}^6C_5 + {}^6C_6\right)$$

$$= \frac{1}{64} \times (15 + 6 + 1)$$

$$= \frac{1}{64} \times 22 = 0.344$$

Example 2 :

The average percentage of failure in a certain examination is 40. What is the probability that out of a group of 6 candidates, at least 4 passed in the examination ?

Solution :

All the trials are independent. The number of pass in the examination may be minimum 4 or 5 or all of them may pass.

$$P = 1 - q$$

$$q = \frac{40}{100} = 0.4$$

$$p = 1 - 0.4 = 0.60$$

The probability of passing 4 or more candidates

i.e., $\quad P(x \geq 4) = P(x = 4) + P(x = 5) + P(x = 6)$

$= P(4) + P(5) + P(6)$

$= {}^6C_4(0.6)^4(0.4)^2 + {}^6C_5(0.6)^5(.4) + {}^6C_6(0.6)^6$

$= (15 \times 0.1296 \times 0.16) + (6 \times 0.07776 \times 0.4) + (0.046656)$

$= 0.311040 + 0.186624 + 0.046656$

$= 0.544320$

Example 3 :

Four coins are tossed simultaneously. What is the probability of getting *(a)* 2 heads and 2 tails (6) at least two heads (c) at least one head.

Solution :

$$P(x) = {}^4C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}$$

(a) Putting x = 2

$$P(2) = {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$$

$$= {}^4C_2\left(\frac{1}{2}\right)^4$$

$$= \frac{3}{8}$$

(b) At least 2 heads mean minimum 2 or 3 or 4 heads. Therefore, the probability :

$$P(\geq 2) = P_2 + P_3 + P_4$$

$$= {}^4C_2\left(\frac{1}{2}\right)^4 + {}^4C_3\left(\frac{1}{2}\right)^4 + {}^4C_4\left(\frac{1}{2}\right)^4$$

$$= \frac{3}{8}+\frac{1}{4}+\frac{1}{16}=\frac{11}{16}$$

(c) P = 1 – P (no head)

$$= 1 - 1 - {}^4C_0\left(\frac{1}{2}\right)^4 = 1-\frac{1}{16}=\frac{15}{16}$$

10.2.2. PROPERTIES OF BINOMIAL DISTRIBUTION

1. Binomial distribution has two parameters – n and p (or q)
2. Mean = np
3. Variance = npq
4. Standard Deviation = $\sqrt{npq}$
5. Binomial distribution is symmetrical if p = q = 0.5

Example 4 :

Five coins are tossed 3,200 times, find the frequencies of the distribution of heads and tails and tabulate the results.

Solution :

p = 0.5 and q = 0.5

Applying binomial distribution, the probability of getting X beads is given by:

$$p(X) = {}^5C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{5-x} = {}^5C_x\left(\frac{1}{2}\right)^5$$

Number of heads (*X*)	$f(x) = 3{,}200 \times {}^5C_x\left(\frac{1}{2}\right)^5$
0	$f(0) = 3{,}200 \times {}^5C_0\left(\frac{1}{2}\right)^5 = 100$
1	$f(1) = 3{,}200 \times {}^5C_1\left(\frac{1}{2}\right)^5 = 500$
2	$f(2) = 3{,}200 \times {}^5C_2\left(\frac{1}{2}\right)^5 = 1000$
3	$f(3) = 3{,}200 \times {}^5C_3\left(\frac{1}{2}\right)^5 = 1000$
4	$f(4) = 3{,}200 \times {}^5C_4\left(\frac{1}{2}\right)^5 = 500$
5	$f(5) = 3{,}200 \times {}^5C_5\left(\frac{1}{2}\right)^5 = 100$
Total	3,200

Example 5 :

A box contains 100 transistors, 20 out of which are defective, 10 are selected for inspection. Indicate what is the probability that

(i) all 10 are defective,

(ii) all 10 are good,

(iii) at least one is defective , and

(iv) at the most 3 are defective ?

Solution:

Let 'X' represent the number of defective transistors selected. Then the possible values of 'X' are 1, 2, 3, ..., 10.

Now

p = *p*(transistor is defective)

$$= \frac{20}{100} = \frac{1}{5}; q = 1 - \frac{1}{5} = \frac{4}{5}$$

Using formula for binomial distribution, the probability of *X* defective transistors is

$p(x) = {}^{10}C_x (1/5)^x (4/5)^{1-x}$

(i) Probability that all 10 are defective is

$$p(10) = {}^{10}C_{10} (1/5)^{10} (4/5)^0 = \frac{1}{5^{10}}$$

(ii) Probability that all 10 are good

$$= 1 - P \text{ (all are defective)} = 1 - \frac{1}{5^{10}}$$

. (iii) Probability that at least one is defective is given by the sum of probabilities,

viz. $p(1) + p(2) + p(3) + \ldots\ldots\ldots + p(10)$

or $1 - p(0) = 1 - {}^{10}C_{10} (1/5)^0 (4/5)^{10} = 1 - (4/5)^{10}$

(iv) Probability of at the most three defective items is

$$\begin{aligned} p(X < 3) &= p(X = 0) + p(X = 1) + (X = 2) + p(X = 3) \\ &= p(0) + p(1) + p(2) + p(3) \\ &= {}^{10}C_0 (1/5)^0 (4/5)^{10} + {}^{10}C_1 (1/5)^1 (4/5)^9 + {}^{10}C_2 (1/5)^2 (4/5)^8 + {}^{10}C_3 (1/5)^3 (4/5)^7 \\ &= 1(.107) + 10(.026) + 45(.0067) + 120(.0016) \\ &= 0.107 + 0.26 + 0.30 + 0.192 = 0.859 \end{aligned}$$

Example 6 :

The incidence of occupational disease in an industry is such that the workmen have a 20% chance of suffering from it. What is the probability that out of six workmen, 4 or more will contact the disease ?

Solution :

Let X represent the number of workers suffering from the disease. Then the possible values of X are 0, 1, 2, ... 6.

$$\begin{aligned} p &= p \text{ (worker suffer from a disease)} \\ &= 20/100 = 1/5 \\ q &= 1 - (1/5) = 4/5, \; n = 6 \end{aligned}$$

Using the formula for binomial distribution, we have

$$p(X) = {}^6C_x (1/5)^x (4/5)^{6-x}$$

the probability that 4 or more workers contact the disease is

$$\begin{aligned} p(X>4) &= p(4) + p(5) + p(6) \\ &= {}^6C_4 (1/5)^4 (4/5)^2 + {}^6C_5 (1/5)^5 (4/5) + {}^6C_6 (1/5)^6 \\ &= \frac{15 \times 16}{15625} + \frac{6 \times 4}{15625} + \frac{1}{15625} = \frac{265}{15625} \\ &= 0.016 \end{aligned}$$

Example 7 :

The probability that an evening college student will graduate is 0.4. Determine the probability that out of 5 students (a) none, (b) one, and (c) atleast one will graduate.

Solution :

$$n = 5 \quad p = 0.4\left(\frac{4}{10}\right);\ q = 0.6\left(\frac{6}{10}\right)$$

$$P(r) = {}^5C_r p^r q^{n-r} = {}^5C_r\left(\frac{4}{10}\right)^r\left(\frac{6}{10}\right)^{n-r}$$

(a) The probability of zero success $= 5C_0\left(\frac{4}{10}\right)^0\left(\frac{6}{10}\right)^5$

$= 1\times\left(\frac{6}{10}\right)^5$

$= 0.078$

(b) The probability of one success $= {}^5C_1\times\left(\frac{4}{10}\right)\times\left(\frac{6}{10}\right)^4$

$= 5 \times 4 \times (6)^4$

$= 0.259$

(c) The probability of atleast one success.

= 1– probability of no success

= 1– 0.078

= 0.922

Example 8 :

12 coins are tossed. What are the probabilities in a single tossing getting :

(1) 9 or more heads,

(2) less than 3 heads,

(3) atleast 8 heads

If the 12 coins are tossed 4096 times, or

(4) how many occasions would you expect these to be :

(a) less than 3 heads,

(b) atleast 2 heads,

(c) exactly 6 heads.

Solution :

(1) The probability of getting 9 or more heads is :

P(9) + P(10) + P(11) + P(12)

$$P(9) = {}^{12}C_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^3$$

$$P(10) = {}^{12}C_{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{2}$$

$$P(11) = {}^{12}C_{11}\left(\frac{1}{2}\right)^{11}\left(\frac{1}{2}\right)$$

$$P(12) = {}^{12}C_{12}\left(\frac{1}{2}\right)^{12}$$

$$P(\geq 9) = {}^{12}C_{9}\left(\frac{1}{2}\right)^{9}\left(\frac{1}{2}\right)^{3} + {}^{12}C_{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{2} + {}^{12}C_{11}\left(\frac{1}{2}\right)^{11}\left(\frac{1}{2}\right) + {}^{12}C_{12}\left(\frac{1}{2}\right)^{12}$$

$$= \left\{\left(\frac{1}{2}\right)^{12}\left({}^{12}C_{9} + {}^{12}C_{10} + {}^{12}C_{11} + {}^{12}C_{12}\right)\right\}$$

$$= \frac{1}{4096}(220 + 66 + 12 + 1)$$

$$= \frac{299}{4096} \text{ or } 7.3\%$$

(2) The probability of getting less than 3 heads:

$$P(>3) = P(0) + P(1) + P(2)$$

$$= {}^{12}C_{0}\left(\frac{1}{2}\right)^{12} + {}^{12}C_{1}\left(\frac{1}{2}\right)^{12} + {}^{12}C_{2}\left(\frac{1}{2}\right)^{12}$$

$$= \left\{\left(\frac{1}{2}\right)^{12}\left({}^{12}C_{0} + {}^{12}C_{1} + {}^{12}C_{2}\right)\right\}$$

$$= \frac{1}{4096}(1 + 12 + 66)$$

$$= \frac{79}{4096}$$

(3) Probability of atleast 8 heads :

Probability of 8 heads = ${}^{12}C_{8}\left(\frac{1}{2}\right)^{12} = \frac{495}{4096} = 0.1208$

Probability of 8 heads $= \frac{299}{4096} = 0.073$

Probability of atleast 8 heads = 0.1208 + 0.073 = 0.1938

(4) *(a)* Probability of less than 3 heads out of 4096 times

$4096 \times \frac{79}{4096}$ =79 Times

(b) probability of atleast 2 heads $= I - P(0) + P(I)$

$$= 1 - {}^{12}C_0\left(\frac{1}{2}\right)^{12} + {}^{12}C_1\left(\frac{1}{2}\right)^{12}$$

$$= 1 - \frac{13}{4096} = \frac{4083}{4096}$$

∴ The number of occasions of getting at least 2 heads in

$$4096 \times \frac{4083}{4096} = \textbf{4083 times}$$

4096 tosses

(c) Probability of 6 heads:

$$P(6) = {}^{12}C_6\left(\frac{1}{2}\right)^{12}$$

$$= \frac{924}{4096}$$

The number of occasions of getting exactly 6 heads =

$$4096 \times \frac{924}{4096}$$

= 924 times

Example 9 :

The mean, of a binomial distribution is 20 and standard deviation is 4. Find out *n*, *p* and *q*.

Solution:

Mean = 20 (np)

Standard Deviation = 4

$$\sqrt{npq} = 4$$

$$npq = 16$$

$$q = \frac{npq}{np} = \frac{16}{20} \text{ or } 0.8$$

$$p = 1 - 0.8 = .2$$

$$n = \frac{20}{0.2} = 100$$

Example 10 :

Obtain the binomial distribution for which mean is 10 and the variance is 5.

Solution :

The mean of binomial distribution $m = np = 10$

The variance= npq = 5

$$\therefore \quad q = \frac{npq}{np} = \frac{5}{10} = 0.5$$

$$p = 1 - q \quad i.e.,\ 1 - 0 = 0.5$$

$$\therefore \quad np = 10\ i.e.,\ n \times 0.5 = 10$$

$$n = \frac{10}{0.5} = 20$$

Therefore the required binomial distribution is

$$(p + q)^n = (0.5 + 0.5)^{20} \text{ or } \left(\frac{1}{2}+\frac{1}{2}\right)^{20}$$

Example 11 :

Obtain the binomial distribution for which the mean is 20 and the variance is 15.

Solution :

The variance= npq = 15

Mean = np = 20

$$\therefore \quad q = \frac{npq}{np} = \frac{15}{20} = \frac{3}{4}$$

$$\therefore \quad q = 1-\frac{3}{4} = \frac{1}{4}$$

$$np = 20$$

i.e. $n\times\frac{1}{4} = 20$

$$\therefore \quad n = \frac{20}{1/4} = \frac{20\times 4}{1} = 80$$

The binomial distribution is :

$$(p + q)^n = \left(\frac{1}{4}+\frac{3}{4}\right)^{80}$$

Fitting of Binomial Distribution

The probability of 0, 1, 2, 3 success would be obtained by the expansion of $(q+p)^n$. Suppose this experiment is repeated for *N* times, then the frequency of *r* success is;

$$N \times P(r) = N \times {}^nC_r q^{n-r} p^r$$

Putting r = 0, 1, 2....n, we can get the expected or theoretical frequencies of the binomial distribution as follows :

Number of Success(r)	Expected or theoretical frequency
0	$\frac{NP(r)}{Nq^n}$
1	$N\,{}^nC_1\,q^{n-1}\,p$
2	$N\,{}^nC_2\,q^{n-2}\,p^2$
r	$N\,{}^nC_r\,q^{n-r}\,p^r$
n	$N\,p^n$

Example 12 :

8 coins are tossed at a time, 256 times.. Find the expected frequencies of success (getting a. head) and tabulate the result obtained.

Solution :

$$p=\frac{1}{2};q=\frac{1}{2};n=8:N=256$$

The probability of success r times in n -trials is given by ${}^nC_r p^r q^{n-r}$

(or) $P_{(r)} = {}^nC_r p^r q^{n-r}$

$$= {}^nC_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{8-r}$$

$$= {}^8C_r\left(\frac{1}{2}\right)^8$$

Frequencies of 0, 1, 2, ... 8 successes are :

Success		Expected Frequency
0	$256\left(\frac{1}{256}\times {}^8C_0\right)$	1
1	$256\left(\frac{1}{256}\times {}^8C_1\right)$	8
2	$256\left(\frac{1}{256}\times {}^8C_2\right)$	28
3	$256\left(\frac{1}{256}\times {}^8C_3\right)$	56
4	$256\left(\frac{1}{256}\times {}^8C_4\right)$	70

5	$256\left(\frac{1}{256}\times {}^8C_5\right)$	56
6	$256\left(\frac{1}{256}\times {}^8C_6\right)$	28
7	$256\left(\frac{1}{256}\times {}^8C_7\right)$	8
8	$256\left(\frac{1}{256}\times {}^8C_8\right)$	1

10.3 POISSON DISTRIBUTION

Poisson distribution was derived in 1837 by a French mathematician Simeon *D* Poisson (1731-1840). In binomial distribution, the values of *p* and *q* and *n* are given. There is a certainty of the total number of events; in other words, we know the number of times an event does occur and also the times an event does not occur, in binomial distribution. But there are cases where p is very small and *n* is very large, then calculation involved will be long. Such cases will arise in connection with rare events, for example.

1. Persons killed in road accidents.
2. The number of defective articles produced by a quality machine,
3. The number of mistakes committed by a good typist, per page.
4. The number of persons dying due to rare disease or snake bite etc.
5. The number of accidental deaths by falling from trees or roofs etc.

In all these cases we know the number of times an event happened but not how many times it does not occur. Events of these types are further illustrated below :

1. It is possible to count the number of people who died accidently by falling from trees or roofs, but we do not know how many people did not die by these accidents.
2. It is possible to know or to count the number of earth quakes that occurred in an area during a particular period of time, but it is, more or less, impossible to tell as to how many times the earth quakes did not occur.
3. It is possible to count the number of goals scored in a foot-ball match but cannot know the number of goals that could have been but not scored.
4. It is possible to count the lightning flash by a thunderstorm but it is impossible to count as to how many times, the lightning did not flash etc.

Thus *n,* the total of trials in regard to a given event is not known, the binomial distribution is inapplicable, Poisson distribution is made use of in such cases where *p* is very Small. We mean that the chance of occurrence of that event is very small. The occurrence of such events is not haphazard. Their behaviour can also be explained by mathematical law. Poisson distribution may be obtained as a limiting case of binomial distribution. When *p* becomes very small and *n* is large, Poisson distribution may be obtained as a limiting case of binomial probability distribution, under the following conditions :

1. *p,* successes, approaches zero ($p \to 0$)
2. $np = m$ is finite.

The poisson distribution is a discrete probability distribution. This distribution is useful in such cases where the value of *p* is very small and the value of *n* is very large. Poisson distribution is a limited form of binomial distribution as n moves towards infinity and p moves towards zero and np or mean remains constant. That is, a poisson distribution may be expected in cases where the chance of any individual event being a success is small.

The Poisson distribution of the probabilities of Occurrence of various rare events (successes) 0, 1, 2, ... given below:

Number of success	Probabilities p (X)
0	e^{-m}
1	me^{-m}
2	$\frac{m^2e^{-m}}{2!}$
3	$\frac{m^3e^{-m}}{3!}$
r	$\frac{m^re^{-m}}{r!}$
n	$\frac{m^ne^{-m}}{n!}$

e = 2.71828

m = average number of occerence of given distribution

The Poisson distribution is a discrete distribution with a parameter m. The various constants are :

1. Mean $= m = p$
2. Stanlald Deviation $= \sqrt{m}$
3. Variance = m

Example 13 :

A book contains 100 misprints distributed randomly throughout its 100 pages. What is the probability that a page observed at random contains atleast two misprints. Assume Poisson Distribution.

Solution:

$$m = \frac{\text{Total Number of misprints}}{\text{Total number of page}} = \frac{100}{100} = 1$$

Probability that a page contain at least two misprints

$$P(r \geq 2) = 1 - [p(0) + p(1)]$$

$$p(r) = \frac{m^r e^{-m}}{r!}$$

$$p(0) = \frac{1^0 e^{-1}}{0!} = e^{-1} = \frac{1}{e} = \frac{1}{2.7183}$$

$$p(1) = \frac{1^1e^{-1}}{1!} = \frac{1^1e^{-1}}{1!} = e^{-1} = \frac{1}{e} = \frac{1}{2.7188}$$

$$p(0) + p(1) = \frac{1}{2.7183} + \frac{1}{2.7183} = .736$$

$$1 - [p(0)+p(1)]$$

$$= 1 - 0.736 = 0.264$$

Example 14 :

If the mean of a Poisson distribution's 4, find (1) S.D. (4) m_3 (5) m_4

Solution :

$m = 4$

1. S.D. = $\sqrt{m} = \sqrt{4} = 2$
2. $\mu_3 = m = 4$
3. $\mu_4 = m + 3m^2 = 4 + 48 = 52$

Example 15 :

Find the probability that at most 5 defective bolts will be found in a box of 200 bolts, if it is known that 2% of such bolts are expected to be defective. (e^{-4}=0.0183)

Solution : m = n × p

= 200 × .02

= 4

$P(o) = P_{(0)}+P_{(1)}+P_{(2)}+P_{(3)}+P_{(4)}+P_{(5)}$

$$= e^{-4}\left(1+4+\frac{4^2}{2!}+\frac{4^3}{3!}+\frac{4^4}{4!}+\frac{4^5}{5!}\right)$$

$= e^{-4}(1+4+8+10.67+8.53)$

$$= 1-\left[\frac{1}{e}+\frac{1}{e}\right]+1=-1\left[\frac{1}{2.718}+\frac{1}{2.178}\right]$$

= 1 – 0.736 = 0.264

Example 16 :

One fifth per cent of the blades produced by a blade manufacturing factory turn out to be defective. The blades are supplied in packets of 10. Use poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 100,000 packets.

(Given e^{02} = .9802)

Solution :

Here $p = \frac{1}{500}, n=10$

$$m = np = \frac{1}{500}\times 10 = 0.02$$

Using the formula for Poisson distribution, the probability of x defective blades is

$$p(x) = \frac{e^{0.02}(0.02)^x}{x!}$$

The frequencies of 0, 1, 2, 3 defective blades given by

$$f(x) = 1{,}00{,}000x\frac{e^{-0.02}(0.20)^x}{X!}$$

Number of packets with no defective blade

$= 1{,}00{,}000 \times e^{-0.02} = 1{,}00{,}000 \times 0.9802$

$= 98{,}020$

Number of packets with one defective blades

$$P_{(1)} = \frac{e^{-0.02}(.02)^x}{\underline{|}!}$$

$= 1{,}00{,}000 \times e^{-0.02} \times 0.02$

$= 98{,}020 \times .02$

$$= 98{,}020 \times \frac{2}{100}$$

$= 1960.4$

$= 1960$

Number of packets with two defective blades is

$$= 1{,}00{,}000 \times e^{-0.02} \times \frac{(0.02)}{2}$$

$= 98020 \times .0002$

$= 19.6040$

$= 20$

Example 17 :

It is known from past experience that in a certain plant there are on the average 4 industrial accidents per month. Find the probability that in a given year there will be less than 4 aecident. Assume Poisson distribution.

Solution :

$m = 4$

$$p(x = r) = \frac{e^{-m}m^r}{r!} = \frac{e^{-4}4^r}{r!}$$

The required probability that there will be less than 4 accidents is given as :

$p(x<4) = p(x = 0) + p(x = 1) + p(x = 2) + p(x = 3)$

$$= e^{-4}\left(1+4+\frac{4^2}{2!}+\frac{4^3}{3!}\right)$$

$= e^{-4}$ (1+4+8+10.67)

$= e^{-4\times 23.67}$

= 0.01832 × 23.67

= 0.4336

Example 18 :

Comment on the following :

For a Poisson Distribution, Mean = 8 and variance = 7

Solution :

The given statement is wrong, because for a. Poisson Distribution

mean and variance are equal.

10.3.1. Fitting a Poisson Distribution

When we want to fit a Poisson Distribution to a given frequency distribution, first we have to find out the arithmetic mean of the given data *i.e.*, X=m When *m* is known the other values can be found out easily. This is clear from the following illustration :

$$N(P_0) = Ne^{-m}$$

$$N(P_1) = N(Po)\times\frac{m}{1}$$

$$N(P_2) = N(P_1)\times\frac{m}{2}$$

$$N(P_3) = N(P_2)\times\frac{m}{3}$$

$N(P_4) = N(P_3)\times\frac{m}{4}$ and so on

Example 19 :

100 Car Radios are inspected as they come off the production line and number of defects per set is recorded below :

No. of Defects	0	1	2	3	4
No. of sets	79	18	2	1	0

Fit a Poisson Distribution to the above data and calculate the frequencies of 0, 1, 2, 3, .and 4 defects.

($e^{-0.25}$=0.779)

Solution:

Fitting Poisson Distribution

No. of Defective	No. of Sets	fx
0	79	0
1	18	18
2	2	4
3	1	3
4	0	0
	N = 100	$\sum fx = 25$

$$\overline{X} = \frac{25}{100} = 0.25$$

$$e^{-.25} = 0.779 \text{ (Given)}$$

$$NP(0) = Ne^{-m} = 100 \times 0.779 = 77.90$$

$$NP_{(1)} = NP_{(0)} \times \frac{m}{1} = 77.9 \times 0.25 = 19.48$$

$$NP_{(2)} = NP_{(1)} \times \frac{m}{2} = 19.48 \times \frac{0.25}{2} = 2.44$$

$$NP_{(3)} = NP_{(2)} \times \frac{m}{3} = 2.44 \times \frac{0.25}{3} = 0.20$$

$$NP_{(4)} = NP_{(3)} \times \frac{m}{4} = 0.20 \times \frac{0.25}{4} = 0.01$$

Example 20 :

Fit a Poisson Distribution to the following data and calculate the theoretical frequencies :

x:	0	1	2	3	4
f:	123	59	14	3	1

Solution :

x	0	1	2	3	4	
f	123	59	14	3	1	$\sum f = 200$
fx	0	59	28	9	4	$\sum fx = 100$

$$\text{Mean} = \frac{100}{200}$$

$$= 0.5$$

$$NP_{(o)} = Ne^{-m}$$

$$= 200 \times e^{-5}$$

$$= 200 \times .6065 = 121.3$$

Calculation of expected trequencies

x	Frequency (NP(x))	
0	NP(0) = 0	121
1	$NP_{(0)} \times \frac{m}{1} = 121.3 \times .5 = 60.65$	61
2	$NP_{(1)} \times \frac{m}{2} = \frac{60.65 \times .5}{2} = 15.16$	15
3	$NP_{(2)} \times \frac{m}{3} = \frac{15.16 \times .5}{3} = 2.53$	3
4	$NP_{(3)} \times \frac{m}{4} = \frac{2.53 \times .5}{4} = .29$	0
	Total	200

10.4 NORMAL DISTRIBUTION

The Binomial distribution and Poisson distribution discussed above are discrete probability distributions. The normal distribution is highly useful in the field of statistics and is an important continuous probability distribution. The graph of this distribution is called normal curve, a bell-shaped curve extending in both the directions, arriving nearer and nearer to the horizontal axis but never touches it.

The normal distribution was first discovered by the English mathematician De-Moivre (1667-1754) in 1673 to solve the problems in game of chances. Later, it was applied in natural and social science by the French mathematician La Place (1749-1827). Normal distribution is also known as Gaussian distribution (Gaussian Law of Error).

In binomial distribution, which is a discrete distribution as the expression of $N(p + q)^n$ gives the expected frequencies of 0, 1, 2, 3..*N* successes. As *n* gets very large, the problem of computing the frequencies becomes difficult and tedious. This difficult situation is handled by the application of normal curve. This curve not only eliminates tedious computations but also gives close approximation to binomial distribution.

The following illustrations will clear the point.

Example 21 :

We know that when an unbiased coin is tossed 10 times, the probability of getting x heads is:

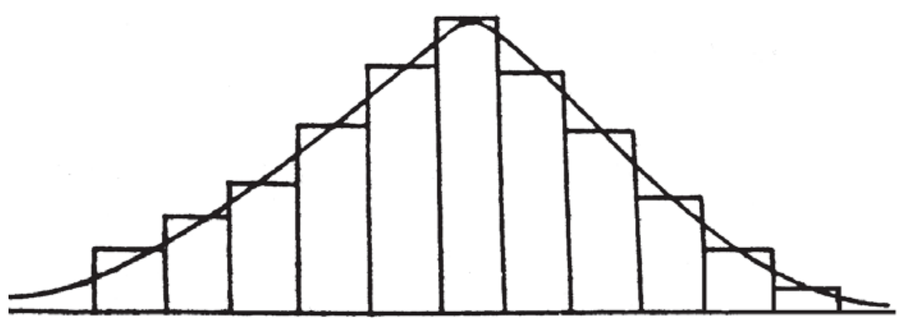
$$p(x) = \frac{n!}{(n-x)!x!}(p)^r (q)^{10-x}$$

x can be 0, 1, 2, 3...10

No. of heads (x)	Probability p(x)
0	1/1024 = 0.0097
1	10/1024 = 0.0098
2	45/1024 =0.0439
3	120/1024 = 0.1172
4	210/1024 =0.2051
5	252/1024 = 0.2461
6	210/1024 = 0.2051
7	120/1024 = 0.1171
8	45/1020 = 0.0439
9	10/1024 = 0.0098
10	1/1024 = 0.0097

$$P(x) = {}^{n}C_{x}p^{x}q^{n-x}$$

Wen may now draw a histogram of the probability distribution, using class frequencies $-\frac{1}{2}$ to $-\frac{1}{2}$, $\frac{1}{2}$ to $1\frac{1}{2}$, $1\frac{1}{2}$ to $2\frac{1}{2}$ ...... $9\frac{1}{2}$ to $10\frac{1}{2}$.



It may be noted that the above figure is symmetrical and bell shaped. It is symmetrical due to the fact, $p = q$, when p is not equal to q, the distribution tends to the form of the normal curve, when n becomes large.

A normal distribution is determined by the parameters-mean and standard deviation. For different values of mean and standard deviation, we get different normal distributions. The area under the normal curve is always taken as unity so as to represent total probability. The area is of great importance in a variety of problems because such an area represents frequency.

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